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Interacting branes, dual branes and dyonic branes: a unifying lagrangian approach in D dimensions

Kurt Lechner and Pieralberto Marchetti

Dipartimento di Fisica, Università degli Studi di Padova, and
Istituto Nazionale di Fisica Nucleare, Sezione di Padova
Via F. Marzolo, 8, 35131 Padova, Italia
E-mail: kurt.lechner@pd.infn.it, pieralberto.marchetti@pd.infn.it

Abstract: This paper presents a general covariant lagrangian framework for the dynamics of a system of closed n-branes and dual $(D-n-4)$-branes in D dimensions, interacting with a dynamical $(n+1)$-form gauge potential. The framework proves sufficiently general to include also a coupling of the branes to (the bosonic sector of) a dynamical supergravity theory. We provide a manifestly Lorentz-invariant and S-duality symmetric lagrangian, involving the $(n+1)$-form gauge potential and its dual $(D-n-3)$-form gauge potential in a symmetric way. The corresponding action depends on generalized Dirac-strings. The requirement of string-independence of the action leads to Dirac-Schwinger quantization conditions for the charges of branes and dual branes, but produces also additional constraints on the possible interactions. It turns out that a system of interacting dyonic branes admits two quantum mechanically inequivalent formulations, involving inequivalent quantization conditions. Asymmetric formulations involving only a single vector potential are also given. For the special cases of dyonic branes in even dimensions known results are easily recovered. As a relevant application of the method we write an effective action which implements the inflow anomaly cancellation mechanism for interacting heterotic strings and five-branes in $D=10$. A consistent realization of this mechanism requires, in fact, dynamical p-form potentials and a systematic introduction of Dirac-strings.

Keywords: p-branes, String Duality, Brane Dynamics in Gauge Theories, Anomalies in Field and String Theories.

Contents

1. Introduction

The unification of the known consistent string theories in a fundamental conjectured M-theory in eleven dimensions relies heavily on the discovery of a set of new extended objects, so called n-branes. Since a basic concept in this unification is duality, electrically charged n-branes in D dimensions are naturally accompanied by magnetically charged dual $(D-n-4)$-branes. The quantum consistency of a theory in which both types of excitations are present relies on a quantization condition for the product of their electric and magnetic charges.

The low energy dynamics of n-branes in string theory is usually described by a Green-Schwarz sigma model in a (super)gravity background, where the background fields, to cope with κ-symmetry, have to satisfy their sourceless *free* equations of motion: one neglects the effect of the backcoupling of the branes on the dynamics of the supergravity fields themselves. Since until now a κ-invariant action describing the dynamics of an *interacting* brane-supergravity system is not known, κ-symmetry can not be used to get information on the nature of this backcoupling.

The present paper opens a different channel for the investigation of the dynamics of such an interacting system, in the absence of fermions, in terms of an action principle. The key-ingredients of the strategy are represented by a systematic introduction of Dirac-strings, i.e. of surfaces whose boundaries are the brane worldvolumes, and of a related manifestly Lorentz invariant action principle describing the interaction between branes and gauge potentials. The introduction of Dirac-strings is, indeed, unavoidable in the following relevant cases: first, when branes and dual branes are simultaneously present in a theory, a well known feature in the case of charges and monopoles in four dimensions; second, when the dynamics of the supergravity theory can not be expressed in terms of the gauge-potential alone which is electrically coupled to the n-brane, but requires also the introduction of its dual potential, [2]. A typical example of the second scenario is represented by the M5-brane in eleven dimensions which couples electrically to the six-form potential A_6 of $N=1$, $D=11$ supergravity. Since the invariant curvature for A_6, given by

$$H_7 = dA_6 + \frac{1}{2} A_3\, dA_3\,,$$

involves its magnetic dual A_3, the supergravity dynamics can not be expressed in terms of A_6 alone. This feature requires a *magnetic* coupling of the M5-brane to A_3, and hence the introduction of Dirac-strings. A similar situation occurs for example for NS5-branes in $D=10$, IIB-supergravity and $D4$-branes in IIA-supergravity.

An appropriate treatment of these situations requires therefore a framework in which potentials and dual potentials appear in a symmetric way. An efficient approach which realizes such a framework is given by the PST-method [3], thanks to its manifest invariance and to its compatibility with all known symmetries. It will therefore be used as a convenient technical ingredient in the present paper.

On the other hand, the presence of Dirac-strings introduces a new consistency requirement on the dynamics: physics has to be independent of the particular choice of the Dirac-string, i.e. the string has to be unobservable. This requirement imposes a quantization condition for the charges, but constrains also the nature of non-minimal interactions between supergravity and n-branes. In this sense Dirac-string independence, as we will see, determines heavily the structure of the backcoupling.

The aim of this paper is to settle a general covariant and flexible lagrangian framework for the description of the dynamics of branes and dual branes interacting with gauge potentials. In the first larger part we will concentrate mainly on the general properties of this system: Lorentz-invariance, effective actions, Dirac quantization conditions, S-duality. Particular attention will be paid to dyons in $4K$ and $4K+2$ dimensions, section 6. As has been noted some time ago [4, 5] the theories of dyons comes in in two different physically *inequivalent* versions, which are governed by inequivalent Dirac-quantization conditions. For each of these theories we will give two *equivalent formulations*; a Schwinger-like formulation in terms of a single vector potential, and a PST-formulation in terms of two vector potentials. It is understood that some of the results of this section have been already derived in [6], in a non-covariant framework.

Section 7 and section 8 are devoted to non-minimal couplings. Particularly interesting are the couplings induced by the inflow anomaly cancellation mechanism. We will work out in detail a simple, but non trivial, example represented by heterotic strings and five-branes interacting with $N=1$, $D=10$ supergravity. The non trivial feature is in this case represented by a modification of the equation of motion for the B_2-potential, induced by the residual anomaly on the five-brane.

Further applications of our framework to the inflow mechanism for M2- and M5-branes in eleven dimensions and NS5-branes in IIA-supergravity are deferred to a future publication [7].

2. Algebra of differential forms

A natural framework for a theory of branes interacting with gauge potentials of higher rank is provided by the language of differential forms. Apart from the notational convenience of this language - one avoids writing indices - it fits also naturally with the extension of Poincaré-duality to "singular" p-form currents, a concept used widely in the text. In this paper space-time is assumed to be topologically trivial so that, in particular, one can make free use of the Poincaré lemma, also for currents.

We summarize here our basic conventions regarding differential forms in D dimensions. Our D-dimensional space-time is endowed with a metric tensor $G_{\mu\nu}(x)$, and the flat metric carries Minkowsky signature $\eta^{\mu\nu}=(1,-1,\ldots,-1)$; the Levi-Civita tensor is characterized by $\varepsilon^{01\ldots D-1}=1$. A p-form is decomposed along its coordinate basis according to

$$\Phi=\frac{1}{p!}\,dx^{\mu_1}\cdots dx^{\mu_p}\,\Phi_{\mu_p\cdots\mu_1}\,.$$

The Hodge-dual of Φ, the $(D-p)$-form

$$*\Phi=\frac{1}{(D-p)!}\,dx^{\mu_1}\cdots dx^{\mu_{D-p}}\,(*\Phi)_{\mu_{D-p}\cdots\mu_1}\,,$$

is defined by

$$(*\Phi)_{\mu_1\cdots\mu_{D-p}} = (-)^{1/2p(p+1)}\,\frac{1}{p!}\,G_{\mu_1\alpha_1}\cdots G_{\mu_{D-p}\alpha_{D-p}}\,\frac{\varepsilon^{\alpha_1\cdots\alpha_{D-p}\nu_1\cdots\nu_p}}{\sqrt{G}}\,\Phi_{\nu_1\cdots\nu_p}\,. \tag{2.1}$$

The sign in the definition of the Hodge-dual is a matter of convention.[1] The advantage of the choice (2.1) is that the $*$-operator squares on a p-form to a sign which is *independent* of p and depends only on the dimension:

$$*^2 = \eta \equiv (-1)^{1/2D(D-1)+1}\,.$$

In even dimensions, $D = 2N$, one has $\eta = (-1)^{N+1}$ and this implies the existence of self-dual (and antiself-dual) N-forms only for odd N, as is well known.

The differential d acts on a form from the right, i.e. $d\,(\Phi_p\Phi_q) = \Phi_p d\Phi_q + (-)^q(d\Phi_p)\Phi_q$, and $d^2 = 0$. Forms in the image of d are called exact and forms in the kernel of d are called closed. Since we work in a topologically trivial space-time all closed forms are also exact. The product between forms will always be an exterior (wedge) product and the wedge symbol $\wedge$ will be omitted. The codifferential is defined by $\delta = *d*$ and lowers the rank of a form by 1. The d'alambertian $\Box = D_\mu G^{\mu\nu} D_\nu$ admits on a p-form the standard decomposition

$$\Box = \eta(-)^D(d\delta + \delta d)\,.$$

For a generic one-form $U = dx^\mu U_\mu$ we indicate with i_U the interior product between the conjugated vector field $U^\mu\partial_\mu$ and a p-form, defined as

$$i_U\Phi = \frac{1}{(p-1)!}\,dx^{\mu_1}\cdots dx^{\mu_{p-1}}U^{\mu_p}\,\Phi_{\mu_p\cdots\mu_1}\,.$$

The operator i_U lowers the rank of a form by 1 and satisfies the same distribution law as d. If U^μ is nowhere lightlike a differential form can be uniquely decomposed in a component along U and a component orthogonal to U. The relevant operatorial identities on p-forms are

$$\begin{aligned}
1 &= (-)^{p+1}\,\frac{1}{U^2}\,(Ui_U + i_U U) = (-)^{p+1}\,\frac{1}{U^2}\,\big(Ui_U + (-)^D\eta * Ui_U*\big)\\
* &= (-)^{p+1}\,\frac{1}{U^2}\,\big(*Ui_U + (-)^D Ui_U*\big)\\
U* &= (-)^p * i_U\,.
\end{aligned} \tag{2.2}$$

3. A covariant action for external brane-sources

In a D-dimensional space-time n-branes are dual to $(D-n-4)$-branes. n-branes carry a natural electric coupling to an $(n+1)$-form gauge potential and a magnetic

[1]The sign chosen in (2.1) differs from the one used in [8]; this leads to some convention depending sign differences between the present paper and [8], when one specializes to the case $D = 4$.

coupling to the dual $(D-n-3)$-form gauge potential; for the dual branes the types of the couplings are reversed. In this picture, which uses two gauge potentials, the dynamics of the gauge degrees of freedom is governed by the Hodge-duality relation between the field strengths associated to the two potentials.

In this section we want describe the dynamics of the gauge degrees of freedom, in the presence of external conserved brane-currents, in terms of a covariant action. The equations of motion which describe this system can be written in the form of generalized Maxwell's equations as

$$dF^1 = J^1 \tag{3.1}$$

$$dF^2 = J^2 \tag{3.2}$$

$$F^1 = *F^2 \Leftrightarrow F^2 = *\eta F^1 \,. \tag{3.3}$$

The curvatures F^I $(I = 1, 2)$ are forms of rank p^I with

$$p^1 = n + 2 \tag{3.4}$$

$$p^2 = D - n - 2 \,, \tag{3.5}$$

and (3.1) and (3.2) imply that the currents, $(p^I + 1)$-forms, are conserved

$$dJ^I = 0 \,. \tag{3.6}$$

For the time being they can also correspond to continuous brane distributions and are not required to be δ-functions on a brane worldvolume. The conservation equations imply the existence of forms C^I such that

$$J^I = dC^I \,. \tag{3.7}$$

They are again forms of rank p^I. Eqs. (3.1) and (3.2) can then be solved introducing potentials A^I, $(p^I - 1)$-forms, such that

$$F^I = dA^I + C^I \,. \tag{3.8}$$

With these definitions of potentials and curvatures the dynamics of the gauge degrees of freedom is described by the duality relation (3.3), which is promoted to an equation of motion.

A Lorentz-invariant action which gives rise to (3.3) can be constructed using a method introduced by Pasti, Sorokin and Tonin (PST) [3]. It requires the introduction of a single scalar auxiliary field $a(x)$ and of the one-form

$$v = \frac{da}{\sqrt{-G^{\mu\nu}\partial_\mu a\,\partial_\nu a}} \equiv dx^\mu v_\mu \,,$$

whose components satisfy $v^2 = v^\mu v_\mu = -1$. In this framework the problem of Lorentz-invariance is converted in the requirement that the field a becomes auxiliary (non propagating). In the PST-approach the auxiliary nature of a relies on the particular symmetries of the PST-action; one of them allows to fix a to an arbitrary value. In this sense the PST-symmetries represent nothing else then the requirement of Lorentz-invariance, and they constrain the form of the action in any dimensions.

The action can be conveniently written in terms of the $(p^I - 1)$-forms

$$
\begin{aligned}
f^1 &\equiv i_v(F^1 - *F^2) && (3.9)\\
f^2 &\equiv i_v(F^2 - *\eta F^1)\,, && (3.10)
\end{aligned}
$$

which parametrize the decomposition of the duality-relation (3.3), associated to v (see (2.2) with $U = v$):

$$
F^1 - *F^2 = (-)^n \left(v f^1 + (-)^{D+1} * v f^2\right). \tag{3.11}
$$

We write the PST-action, as the integral of a D-form, in three different ways,

$$
\begin{aligned}
I_0[A, C, a] &= \int \left[\frac{1}{2}\left(F^1 * F^1 + (-)^n f^1 * f^1\right) - \eta\left(dA^1C^2 + \frac{1}{2}C^1C^2\right)\right]\\
&= \eta(-)^{nD+n+1} \int \left[\frac{1}{2}\left(F^2 * F^2 + (-)^{D+n} f^2 * f^2\right) - \left(dA^2C^1 + \frac{1}{2}C^2C^1\right)\right]\\
&= \frac{1}{2}\int \left[F\,\mathcal{P}(v)\,F + \eta\left(C^1dA^2 - dA^1C^2\right)\right].
\end{aligned} \tag{3.12}
$$

It is understood that the dynamical variables are A^1, A^2 and a. The curvatures are defined as in (3.8) and with F we indicate the ordered couple (F^1, F^2), with form degrees (p^1, p^2). The operator $\mathcal{P}(v)$ sends a couple of (p^1, p^2)-forms in a couple of (p^2, p^1)-forms, such that $F\,\mathcal{P}(v)\,F$ is a D-form. It is defined as

$$
\begin{aligned}
\mathcal{P}(v) = &\begin{pmatrix} (-1)^{D+n} & 0 \\ 0 & \eta(-)^{nD+1} \end{pmatrix} v i_v * + \begin{pmatrix} 0 & \eta(-)^{D+n+1} \\ \eta(-)^{nD} & 0 \end{pmatrix} v i_v + \\
&+ \frac{1}{2}\begin{pmatrix} 0 & \eta \\ \eta(-)^{nD+n+1} & 0 \end{pmatrix}.
\end{aligned} \tag{3.13}
$$

This operator is symmetric in the sense that for two couples F and G of (p^1, p^2)-forms one has $F\,\mathcal{P}(v)\,G = G\,\mathcal{P}(v)\,F$.

The three different expressions of the action can be obtained making repeated use of the identities (2.2) and of $*^2 = \eta$. The first line of (3.12) privileges the potential A^1. If one drops the term $f^1 * f^1$, proportional to the square of the duality relation (3.3), then one obtains a formulation in terms of only the single potential A^1. The corresponding action generalizes the one given by Schwinger [4] for the dynamics of dyons in four dimensional electrodynamics.

The second line privileges the potential A^2 and can be obtained from the first line with the formal substitutions $1 \leftrightarrow 2$, $n \leftrightarrow D - n - 4$. The additional factor of

η comes from the η which appears in the dual duality relation $F^2 - \eta * F^1 = 0$.[2] In the third line the potentials A^1 and A^2 appear in a symmetric way. This form will appear very useful for displaying the S-duality properties of the action and for establishing the duality-symmetry groups in the case of dyonic branes.

As we will see, at the quantum level the theory of interacting dyons comes in in two *inequivalent* versions, while the theory of interacting branes and dual branes admits, actually, only a single version. To display from the very beginning the difference between the two versions, which we call respectively "symmetric" and "asymmetric", we illustrate them here also for the case of branes and dual branes. The action we wrote above corresponds to the "symmetric" theory, in the sense that it is invariant under the formal interchange $1 \leftrightarrow 2$, as explained above. The "asymmetric" theory is obtained by adding to, or subtracting from, I_0 the term $1/2\eta \int C^1 C^2$. In the first case one cancels in the first line in (3.12) the last term, in the second case one cancels the last term in the second line. In the following we will show that the two choices, obtained by subtracting or adding this term, lead in *any* case to equivalent quantum theories. For definiteness we choose here for the action of the asymmetric theory

$$\widetilde{I}_0 \equiv I_0 + \frac{1}{2}\eta \int C^1 C^2 \,. \tag{3.14}$$

Since the added term is independent from A^I, it is clear that I_0 and $\widetilde{I}_0$ lead to the same equations of motion for the gauge potentials. We will also see that in the case of *dynamical* branes the equations of motion for the branes will be identical too. The inequivalence between the symmetric and asymmetric theories will arise in the *quantum* theory of dyons, as we will see below; in particular, for the two theories we will obtain different quantization conditions for the charges.

As we mentioned above, a one potential formulation can be obtained by dropping in the first line in (3.12) the term $f_1 * f_1$. For the symmetric theory this leads to the Schwinger-like action, in terms of A_1,

$$I_0 \to I_{\text{Schwinger}} = \int \left[\frac{1}{2}F^1 * F^1 - \eta dA^1 C^2 - \frac{1}{2}\eta \int C^1 C^2\right].$$

For the asymmetric theory one has

$$\widetilde{I}_0 \to \widetilde{I}_{\text{Schwinger}} = \int \left[\frac{1}{2}F^1 * F^1 - \eta dA^1 C^2\right].$$

F^1 satisfies the Bianchi identity $dF^1 = J^1$ and the equation of motion for A^1 is $d(*\eta F^1) = J^2$. These are just the generalized Maxwell equations (3.1)–(3.3). When-

[2]For notational convenience the overall sign in I_0 has been chosen as $+1$. To obtain a positive definite energy the formulae in (3.12) have to be supplied by an additional overall sign given by $\gamma(n) \equiv \eta(-1)^{1/2n(n+1)+D+n+1}$. This ensures positiveness of the term $F^1 * F^1$ in the first line. The positiveness of the term $F^2 * F^2$ in the second line is then guaranteed by the identity $\gamma(n)\eta(-1)^{nD+n+1} = \gamma(D-n-4)$.

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ever there is no need to introduce both potentials, this form of the action for the theory is the most convenient and simple one. We would like to stress that the formulations in terms of a single vector potential and the ones in terms of two vector potentials a la PST are completely equivalent. When the dynamics is described by the simple set (3.1)–(3.3) both options are reliable: the PST-formulation appears more convenient for the description of the symmetric theory (manifest duality invariance), while the Schwinger-formulation appears more convenient for the asymmetric theory. When the dynamics is more complicated (e.g. self-interactions of the potential, chiral bosons, presence of Chern-Simons terms) the PST-formulations might be unavoidable. For definiteness in what follows we use the PST-formulation.

For completeness we illustrate now briefly the structure of the PST-symmetries. All signs and relative coefficients in the action I_0 are indeed uniquely determined by those symmetries. The corresponding transformations are given by

$$\delta A^I = d\Lambda^I \tag{3.15}$$

$$\delta A^I = \Phi^I \, da \tag{3.16}$$

$$\delta A^I = -\frac{\varphi}{\sqrt{-(\partial a)^2}}\, f^I\,, \qquad \delta a = \varphi\,. \tag{3.17}$$

The transformation parameters are the (p^I-2)-forms Λ^I and Φ^I, and the single scalar φ. We relegate the proof of the invariance of I_0 under these transformations to the appendix. The transformations in (3.15) are just ordinary gauge transformations for the potentials A^I, and (3.17) states that the field a is a non propagating auxiliary field which can be shifted to any value. The transformations (3.16) allow to reduce the second order equation of motion for the gauge fields to the first order duality relation (3.3). To see this we write the equations of motion for A and a, derived in the appendix,

$$d\,(vf^I) = 0 \tag{3.18}$$

$$d\left(\frac{v}{\sqrt{-(\partial a)^2}}\, f^1 f^2\right) = 0\,. \tag{3.19}$$

The equation of motion for a (3.19) is a consequence of the A^I-equations (3.18) and contains no dynamical information. The general solution of (3.18) is $vf^I = da\, d\widetilde{\Phi^I}$, for some (p^I-2)-forms $\widetilde{\Phi^I}$; it can then be seen [3] that through a transformation (3.16), with $\Phi^I = \widetilde{\Phi^I}$, one can set $f^I = 0$. But, due to the identity (3.11) this is equivalent to (3.3). Identical conclusions hold also for the action $\widetilde{I}_0$ of the asymmetric theory.

Until now we ignored an ambiguity which is associated to the choice of the C-fields. From their definition (3.7) it is clear that they are defined only modulo exact forms. Since the curvatures should not be affected by this ambiguity, one has to consider the combined finite transformations

$$\begin{aligned} \Delta C^I &= dW^I\,, \\ \Delta A^I &= -W^I\,, \end{aligned} \tag{3.20}$$

where the W^I are $(p^I - 1)$-forms. Whereas the equations of motion for the gauge fields (3.18) are invariant under these transformations the action is not. We call the transformation of the action under (3.20) "Dirac-anomaly", for a reason that will become clear in the next section. For the symmetric and asymmetric theories respectively, the finite Dirac-anomaly (i.e. for finite variations) can be easily computed to be [3]

$$A_D \equiv \Delta I_0 = \frac{1}{2}\eta(-)^{D+n+1}\int \left(W^1 J^2 + J^1 W^2\right) \tag{3.21}$$

$$\widetilde{A}_D \equiv \Delta\widetilde{I}_0 = \eta(-)^{D+n+1}\int W^1 J^2\,. \tag{3.22}$$

In the presence of dynamical branes the transformations (3.20) will correspond to a generalized change of Dirac-strings, and the vanishing of the (exponentiated) Dirac-anomaly will trigger the consistency of the quantum dynamics of these extended objects and impose quantization conditions for the charges. The difference between the formulae (3.21) and (3.22) reflects the inequivalence between the symmetric and asymmetric theories, mentioned above.

The Dirac-anomalies associated to the Schwinger-like actions are also given by (3.21), (3.22); this is a trivial consequence of the equivalence between the PST- and Schwinger-formulations.

One more word is in order for what concerns the identification of the observable degrees of freedom carried by the gauge fields. A look on the structure of the PST-symmetries, more precisely on (3.16), shows that the curvatures F^I are not invariant objects, even if the equations of motion (3.18) are satisfied; therefore, they do not correspond to observable quantities. The observable curvatures, which can indeed be seen to be invariant under (3.15)–(3.17) if (3.18) holds, are given by

$$K^1 = F^1 + (-)^{n+1} v f^1 \tag{3.23}$$

$$K^2 = F^2 + (-)^{D+n+1} v f^2\,. \tag{3.24}$$

They are also invariant under (3.20) and, moreover, they satisfy identically the duality relation

$$K^1 = *K^2\,.$$

Once the symmetry (3.16) is fixed as above, i.e. $f^I = 0$, one has $K^I = F^I$. Notice that in terms of the K^I-tensors the equations of motion (3.18) can be written also

[3] Considering e.g. the third line in (3.12) one sees that the first term is trivially invariant, while in the second and third terms only the variations of A^I contribute.

as

$$dK^I = J^I \,. \tag{3.25}$$

Therefore, the K^I's satisfy the same equations as the F^I's, i.e. (3.1)–(3.3). For the F^I's (3.1) and (3.2) are identities and the duality relation is an equation of motion. For the K^I's the situation is reversed: they satisfy the duality relation identically and (3.25) is their equation of motion.

To illustrate the implementation of Lorentz-invariance in the present framework we derive now the current-current effective action. The knowledge of this effective action will display also more clearly the difference between symmetric and asymmetric theories. For the symmetric theory it is defined by the (normalized) functional integral over A^I and a of the exponentiated classical action,

$$e^{i\Gamma[C]} = \frac{\int \{\mathcal{D}A\}\,\{\mathcal{D}a\}\, e^{iI_0[A,C,a]}}{\int \{\mathcal{D}A\}\,\{\mathcal{D}a\}\, e^{iI_0[A,0,a]}} \,. \tag{3.26}$$

Appropriate gauge fixings of the PST-symmetries are understood, in particular the insertion of a δ-function $\delta(a - a_0)$ for the symmetry (3.17), where $a_0(x)$ is an arbitrary scalar gauge fixing function. The requirement of Lorentz-invariance translates here in the requirement that $\Gamma[C]$ is independent of the (unphysical) function a_0. But this is ensured by the PST-symmetry (3.17) of the action I_0. An explicit evaluation of the right hand side of (3.26) (a sketch of the computation is given in the appendix 8) leads indeed to the a_0-independent result

$$\Gamma[C] = (-)^D \frac{1}{2} \int \left(J \frac{*}{\Box} \mathcal{M}\, J + J \frac{\delta}{\Box} \mathcal{N}\, C \right) . \tag{3.27}$$

Here $J = (J^1, J^2)$, $C = (C^1, C^2)$ and the 2×2 matrices $\mathcal{M}$ and $\mathcal{N}$ are respectively diagonal and off-diagonal,

$$\mathcal{M} = \begin{pmatrix} (-1)^{D+n} & 0 \\ 0 & \eta(-1)^{nD+1} \end{pmatrix} \tag{3.28}$$

$$\mathcal{N} = \begin{pmatrix} 0 & (-1)^{D+n+1} \\ (-1)^{nD} & 0 \end{pmatrix} . \tag{3.29}$$

For the asymmetric theory the effective action becomes

$$\begin{aligned} \widetilde{\Gamma}[C] &= (-)^D \int \left(\frac{1}{2} J \frac{*}{\Box} \mathcal{M}\, J + (-)^{nD} J^2 \frac{\delta}{\Box} C^1 \right) \\ &= \Gamma[C] + \frac{1}{2}\eta \int C^1 C^2 \,. \end{aligned} \tag{3.30}$$

By construction the difference between $\widetilde{\Gamma}$ and Γ equals the difference between $\widetilde{I}_0$ and I_0. The diagonal terms in Γ and $\widetilde{\Gamma}$, which involve only the currents J, coincide and represent the Coulomb-like electric-electric and magnetic-magnetic interactions.

The mixed terms, representing the electric-magnetic interactions, involve also the "strings" C and differ by the term $1/2\eta\int C^1C^2$. The Dirac-anomalies carried by the effective actions, defined as their variations under $C^I \to C^I + dW^I$, are, by construction, given again by (3.21) and (3.22). The technical tool involved in checking these assertions on the effective actions is the standard Hodge decomposition of the d'alambertian, in the form

$$1 = \eta(-)^D(d\delta + \delta d)\cdot\frac{1}{\Box}\,.$$

More explicit expressions for the effective actions will be given for the particular cases considered below.

We would like to stress that the effective actions obtained from the Schwinger formulations, with a single vector potential, coincide with the ones given above; this is again a consequence of the equivalence of the two formulations.

4. Dynamical n-branes and dual $(D-n-4)$-branes

Until now we took the currents J^I as arbitrary conserved external currents. The physically interesting case corresponds to currents which are associated to a certain number of extended charged objects, i.e. charged branes. The aim of the present section is to describe the interaction of the potentials A^I with a set of dynamical branes and dual branes in terms of a covariant action. The essential new feature of this situation is represented by the requirement of the vanishing of the (exponentiated) Dirac-anomaly, leading eventually to charge quantization. The analysis of the Dirac-anomaly can be carried out most easily using the concept of Poincaré-duality between hypersurfaces and "p-currents". So it will be briefly reviewed below.

We consider a system of a certain number N_2 of closed n-branes, with charges e^2_r and tensions T^2_r $(r = 1,\ldots,N_2)$, and of N_1 closed dual $(D-n-4)$-branes, with charges e^1_r and tensions T^1_r $(r = 1,\ldots,N_1)$.

During time evolution an n-brane sweeps out a boundaryless $(n+1)$-dimensional worldvolume parameterized by $y^\mu(\sigma^i)$, where $i = (0,1\ldots,n)$ and σ^0 indicates the evolution parameter. The dynamics of such an extended object can be described, for example, in terms of the Polyakov-like action

$$I_n[g(\sigma),y(\sigma)] = \frac{T}{2}\int d^{n+1}\sigma\sqrt{g}\left(g^{ij}\partial_i y^\mu\partial_j y^\nu G_{\mu\nu}(y) + (1-n)\right). \tag{4.1}$$

The worldvolume metric g_{ij} is treated as an independent variable and $g \equiv -\mathrm{det}g_{ij}$. Its equation of motion, for $n \neq 1$, leads to the induced metric

$$g_{ij} = \partial_i y^\mu\partial_j y^\nu G_{\mu\nu}(y)\,. \tag{4.2}$$

Substituting it back in (4.1) leads to the (at the classical level equivalent) Nambu-Goto-like action $I_n[y] = T\int d^{n+1}\sigma\sqrt{g}$, where g_{ij} is now the induced metric.

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The action which describes the interaction between the system of branes and the gauge degrees of freedom can then be written as

$$I[A,C,a] = I_0[A,C,a] + \sum_{r=1}^{N_2} I_n^r + \sum_{r=1}^{N_1} I_{D-n-4}^r\,, \tag{4.3}$$

where each of the individual kinetic terms for branes and dual branes is of the form (4.1). The action for the asymmetric theory is given by

$$\widetilde{I} = I + \frac{1}{2}\eta \int C^1 C^2\,. \tag{4.4}$$

To complete the description of the dynamics, and specify the action completely, one has to properly define the C^I-forms in terms of the dynamical variables of the branes. To provide this link it is convenient to rely on the concept of Poincaré-duality, generalized to the space of differential p-forms with distribution-valued components i.e. p-currents. In this space Poincaré-duality associates to every p-dimensional hypersurface Σ_p a $(D-p)$-form Φ_{Σ_p}, such that

$$\int_{\Sigma_p} \Psi_p = \int_{R^D} \Phi_{\Sigma_p} \Psi_p\,, \tag{4.5}$$

for any p-form Ψ_p. The explicit expression for Φ_{Σ_p} is given in the appendix 8. The Poincaré-duality-map respects, in particular, the exact-forms $\leftrightarrow$ boundary-hypersurfaces correspondence, i.e.

$$\Sigma_p \to \Phi_{\Sigma_p} \qquad \text{implies} \tag{4.6}$$

$$\partial\Sigma_p \to d\Phi_{\Sigma_p}\,, \tag{4.7}$$

where ∂ indicates the boundary operator.[4] A basic consequence of Poincaré-duality is that the integral of Φ_{Σ_p} over a generic $(D-p)$-dimensional surface Σ_{D-p} is an integer, counting the intersections with sign of Σ_p with Σ_{D-p}. This implies, in particular, that the integral over R^D of a product of two such forms is also an integer,

$$\int_{R^D} \Phi_{\Sigma_p} \Phi_{\Sigma_{D-p}} = \int_{\Sigma_p} \Phi_{\Sigma_{D-p}} = N\,, \tag{4.8}$$

which counts the number of intersections with sign of Σ_p with Σ_{D-p}. Linear combinations of such p-forms with integer coefficients are called *integer* forms. The property (4.8) holds also for generic integer forms: the integral of a product of two integer forms is an integer, whenever the integral is well defined.

The forms C^I, whose defining property is $dC^I = J^I$ (see (3.7)), can be determined as follows. The worldvolume of say the r-th n-brane is a closed $(n+1)$-dimensional

[4] See also [8]; for a mathematically precise formulation, involving chains, see [9].

hypersurface Σ_r^2, parameterized by $y_r^\mu(\sigma)$. Its Poincaré-dual is a closed integer $(D-n-1)$-form; we call it J_r^2, $dJ_r^2=0$. The total current J^2, appearing in (3.2), is then given by

$$J^2=\sum_{r=1}^{N_2} e_r^2 J_r^2\,.$$

Since Σ_r^2 is a closed $(n+1)$-dimensional hypersurface, due to the Poincaré lemma for currents [9], there exists an $(n+2)$-dimensional hypersurface Ω_r^2, a generalized "Dirac-string", such that $\partial\Omega_r^2=\Sigma_r^2$. Consequently, the Poincaré-dual of Ω_r^2 is a $(D-n-2)$-form, we call it C_r^2, satisfying $J_r^2=dC_r^2$. Therefore,

$$C^2=\sum_{r=1}^{N_2} e_r^2 C_r^2\,.$$

Since the same analysis applies also to the dual $(D-n-4)$-branes, we have

$$\begin{aligned} J^I &= \sum_{r=1}^{N_I} e_r^I J_r^I\,,\\ C^I &= \sum_{r=1}^{N_I} e_r^I C_r^I\,. \end{aligned} \tag{4.9}$$

However, the hypersurfaces Ω_r^I are not uniquely determined since, in general, there exist infinitely many hypersurfaces whose boundaries are Σ_r^I. But under a Dirac-string change $\Omega_r^I\to\widehat{\Omega}_r^I$, we have $\widehat{\Omega}_r^I-\Omega_r^I=\partial\Lambda_r^I$, for some hypersurface Λ_r^I. Under such a string change the C-fields vary as

$$\begin{aligned} \widehat{C}^I-C^I &= dW^I\,,\\ W^I &\equiv \sum_{r=1}^{N_I} e_r^I W_r^I\,, \end{aligned} \tag{4.10}$$

where the W_r^I, (p^I-1)-forms, are Poincaré-dual to Λ_r^I. These are precisely the transformations which generate the Dirac-anomaly, see (3.20).

For what follows it is important to realize that the forms J_r^I, C_r^I and W_r^I are all *integer* forms.

With the above determination of the C^I-fields, the action for the interacting system given in (4.3) becomes, actually, a functional of the *strings* C^I, and not only of the worldvolumes, parameterized by $(y^I)_r^\mu(\sigma)$. But, what needs to be string-independent at the *quantum* level is rather the exponential $\exp(iI)$ then the action itself. By definition, the response of the action under a string change is measured by the Dirac-anomalies . For the symmetric and asymmetric theories respectively, they can be evaluated using (3.21), (3.22), (4.9) and (4.10):

$$A_D = \frac{1}{2}\eta(-)^{D+n+1}\sum_{r,s} e_r^1 e_s^2 \int \left[W_r^1 J_s^2 + J_r^1 W_s^2\right] \tag{4.11}$$

$$\widetilde{A}_D = \eta(-)^{D+n+1}\sum_{r,s} e_r^1 e_s^2 \int W_r^1 J_s^2 \,. \tag{4.12}$$

Quantum consistency constrains these anomalies to be integer multiples of 2π. Since the integrands are products of integer forms, the integrals in the expressions above are (positive or negative) integer numbers, and we obtain as quantization conditions for the charges, respectively for the symmetric and asymmetric theory

$$\frac{1}{2}e_r^1 e_s^2 = 2\pi n_{rs} \tag{4.13}$$

$$e_r^1 e_s^2 = 2\pi \widetilde{n}_{rs}\,, \tag{4.14}$$

for each r and s.

If the charges satisfy the stronger condition (4.13), then the symmetric and asymmetric theories coincide, actually. In this case we have, indeed,

$$\widetilde{I} - I = \frac{1}{2}\eta \int C^1 C^2 = \frac{1}{2}\eta \sum_{r,s} e_r^1 e_s^2 \int C_r^1 C_s^2 = 2\pi m\,,$$

with m integer (here we used the fact that the C_r^I are integer forms, too).

We can conclude that a necessary and sufficient condition for the quantum consistency of a system of interacting branes and dual branes is the (weaker) Dirac-condition (4.14), and that the corresponding theory is described by the (asymmetric) action $\widetilde{I}$. The theory based on the action I, with the consistency condition (4.13), is just a special case of the asymmetric theory.

Hence, in the case of branes/dual branes there is only *one* quantum mechanically consistent theory — as anticipated above — whose dynamics is represented by the asymmetric action $\widetilde{I}$, with the consistency condition (4.14).

We anticipated also that there is only *one* asymmetric theory in that the two actions $\widetilde{I}_\pm = I \pm 1/2\eta \int C^1 C^2$ are equivalent. We see now that the equivalence stems from the fact that, due to (4.14), the difference $\widetilde{I}_+ - \widetilde{I}_- = \eta \int C^1 C^2$ is an integer multiple of 2π.

Once the functional $S \equiv \exp(i\widetilde{I})$ has been established to be string-independent it becomes a functional of only the brane worldvolumes. Therefore, an action principle based on S leads to equations of motion which are automatically string-independent and respect, moreover, the PST-symmetries. All this information restricts notably the form of the equations of motion for the branes, and their derivation is a mere exercise.[5] For the n-branes and $(D-n-4)$-branes one obtains respectively the following equations of motion:

$$\begin{aligned} T_r^2 \,\Box\, y_r^\mu &= \eta(-)^{nD}\, e_r^2\, \frac{\partial y_r^{\nu_1}}{\partial\sigma^0}\cdots\frac{\partial y_r^{\nu_{n+1}}}{\partial\sigma^n}\,(K^1)^\mu{}_{\nu_{n+1}\cdots\nu_1}(y_r) \\ T_r^1 \,\Box\, y_r^\mu &= \eta(-)^{D+n+1}\, e_r^1\, \frac{\partial y_r^{\nu_1}}{\partial\sigma^0}\cdots\frac{\partial y_r^{\nu_{D-n-3}}}{\partial\sigma^{D-n-4}}\,(K^2)^\mu{}_{\nu_{D-n-3}\cdots\nu_1}(y_r)\,. \end{aligned} \tag{4.15}$$

The completely covariant laplacians on the worldvolumes are defined by

$$\Box\, y^\mu = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j y^\mu) + g^{ij}\partial_i y^\alpha \partial_j y^\beta \Gamma^\mu_{\alpha\beta}\,,$$

where the metric g^{ij} is the induced one (4.2), and $\Gamma^\mu_{\alpha\beta}$ is the affine target-space connection, evaluated on the brane.

These are the expected equations of motion. The appearance of the invariant tensors K^I, instead of the F^I, is dictated by the PST-symmetries, as explained above.

5. S-duality

A basic property of the PST-action (3.12) is its invariance under a simultaneous S-duality transformation of the vector potentials A^I.

To prove this statement we give first an equivalent representation for the third line in (3.12). For this purpose we introduce a couple of (dual) gauge potentials Λ^I, with degrees $D-p^I-1$, and a couple of p^I-forms G^I. With these new ingredients we write the equivalent action

$$I_0[G,\Lambda,C,a] \equiv \frac{1}{2}\int (G+C)\mathcal{P}(v)(G+C) + \eta\left(C^1G^2 - G^1C^2\right) + \eta\left(d\Lambda^1 G^2 - G^1 d\Lambda^2\right). \tag{5.1}$$

The equations of motion for Λ^I give $dG^I = 0 \to G^I = dA^I$. Substituting this expression for G^I in (5.1) one is back to the PST-action.

The S-transformed action, $\widehat{I}_0$, is defined by the functional integral over G^I,

$$e^{i\widehat{I}_0[\Lambda,C,a]} \equiv \int \{\mathcal{D}G\}\, e^{iI_0[G,\Lambda,C,a]}\,. \tag{5.2}$$

To evaluate the integral it is convenient to perform the shift $G^I \to G^I - C^I$,

$$e^{i\widehat{I}_0[\Lambda,C,a]} = e^{i\frac{\eta}{2}\int C^1 d\Lambda^2 - d\Lambda^1 C^2} \int \{\mathcal{D}G\}\, e^{i/2\int G\,\mathcal{P}(v)\,G + \eta((d\Lambda^1+C^1)G^2 - G^1(d\Lambda^2+C^2))}\,. \tag{5.3}$$

[5] Some caution is required if one wants to take into account also configurations for which the string associated to a brane intersects the worldvolume of a dual brane; for these configurations the consistent regularization procedure worked out in [8] for dyons in four dimensions applies equally well also in the present case.

The prefactor of the functional integral reproduces already the last term in the third line of (3.12) (with the substitution $A^I \to \Lambda^I$), and the (gaussian) functional integral itself depends only on the combinations $d\Lambda^I + C^I$. To perform it explicitly one has only to know the inverse of the operator $\mathcal{P}(v)$; apart from an overall factor of 4 and of some sign changes the structure of this operator is the same as the one of $\mathcal{P}(v)$. It is given by

$$\begin{aligned}\frac{1}{4}\mathcal{P}^{-1}(v) = &\begin{pmatrix} \eta(-)^n & 0 \\ 0 & (-1)^{nD+n+1} \end{pmatrix} v i_v * + \begin{pmatrix} 0 & \eta(-)^{nD} \\ \eta(-)^{D+n+1} & 0 \end{pmatrix} v i_v + \\ &+ \begin{pmatrix} 0 & \eta(-)^{nD+n+1} \\ \eta & 0 \end{pmatrix} .\end{aligned} \tag{5.4}$$

This leads to the simple result

$$\int \{\mathcal{D}G\}\, e^{i/2\int G\,\mathcal{P}(v)\,G + \eta((d\Lambda^1+C^1)G^2 - G^1(d\Lambda^2+C^2))} = e^{i/2\int (d\Lambda+C)\mathcal{P}(v)(d\Lambda+C)} .$$

Using this in (5.3) we arrive at

$$\widehat{I}_0[\Lambda, C, a] = I_0[\Lambda, C, a] ,$$

i.e. again the PST-action for the couple Λ^I.

6. Particular cases

We discuss in this section some particular relevant cases: chiral bosons, dyons in twice even dimensions and dyons in twice odd dimensions. The actions which govern the dynamics of these objects are obtained by specializing the general action (3.12) to these cases.

6.1 Chiral bosons and self-dual branes in $D = 4K + 2$

We begin by considering gauge potentials with (anti)self-dual field strengths — (anti)chiral bosons — which exist in dimensions $D = 4K + 2$. In this case we have a single $2K$-form gauge potential A and a single $(2K+1)$-form string C. The curvature is also a $(2K+1)$-form, $F = dA + C$, and satisfies as equations of motion the (anti)self-duality relations

$$F = \pm * F , \tag{6.1}$$

for chiral and antichiral bosons respectively. The consistency of these equations is guaranteed by the fact that now

$$*^2 = \eta = +1 .$$

The branes coupled to these vector potentials are $(2K-1)$-branes ($n = 2K-1 =$ odd), and coincide with the dual branes. Correspondingly we have now only a single type of (electric = magnetic) charges

$$e_r^1 = e_r^2 \equiv e_r\,.$$

These branes can couple to chiral or antichiral bosons. The actions for these systems can be obtained by enforcing in (3.12) the identifications

$$\begin{aligned} A^1 &= A\,, \qquad & A^2 &= \pm A\,, \\ C^1 &= C\,, \qquad & C^2 &= \pm C\,, \end{aligned} \tag{6.2}$$

which lead to $F^1 = F$, $F^2 = \pm F$. With these substitutions one obtains, for chiral and antichiral bosons respectively, the actions[6]

$$I_0^{\pm}[A, C, a] = \pm \int F v i_v (F \mp *F) + C dA\,. \tag{6.3}$$

It is important, but easy, to realize that the above identifications are compatible with the PST-symmetries. Therefore, a is again an auxiliary field and the equation of motion for A is $d(vf^{\pm}) = 0$, where $f^{\pm} \equiv i_v(F \mp *F)$. With the usual gauge fixing of the symmetry (3.16) for the single potential A, it reduces just to (6.1).

Since in the present case C is an odd form the term $\int CC$ is identically zero and we have only a single type of theories. The Dirac-anomalies can be obtained inserting (6.2) in (3.21) (or (3.22)) with the result

$$\mathcal{A}_D^{\pm} = \Delta I_0^{\pm} = \pm \int W J\,. \tag{6.4}$$

Clearly they can be obtained directly also from (6.3). The variations are here $\Delta C = dW$, $\Delta A = -W$, and $J \equiv dC$.

If we have a number N of dynamical branes with charges e_r^+ (e_r^-) coupled to chiral (antichiral) bosons, the expressions for J, C and W are, as in the previous section, $J = \sum_r e_r^{\pm} J_r$, $C = \sum_r e_r^{\pm} C_r$, $W = \sum_r e_r^{\pm} W_r$, with J_r, C_r and J_r integer forms. The Dirac-anomalies become then

$$\mathcal{A}_D^{\pm} = \pm \sum_{r,s} e_r^{\pm} e_s^{\pm} \int W_r J_s\,,$$

and the quantization condition is of the Dirac-type

$$e_r^{\pm} e_s^{\pm} = 2\pi n_{rs}\,, \tag{6.5}$$

[6]It is understood that here and in the cases treated below eventually one has to add also the kinetic terms for the branes $\sum_r T_r \int d^{n+1}\sigma \sqrt{g_r}$.

for each r and s. A class of solutions[7] of these conditions is given by

$$e_r^{\pm} = \sqrt{2\pi L} \cdot n_r \,, \tag{6.6}$$

where L is a fixed integer and the n_r are integers, too.

The effective actions amount now to

$$\Gamma_{\pm}[C] = \int \left(-J \frac{*}{\Box} J \pm J \frac{\delta}{\Box} C \right) = \sum_{r,s} e_r^{\pm} e_s^{\pm} \int \left(-J_r \frac{*}{\Box} J_s \pm J_r \frac{\delta}{\Box} C_s \right). \tag{6.7}$$

The first term represents the standard Coulomb interactions between the branes; the second term represents the mixed interactions which we called previously electric-magnetic interactions. The striking feature in this case is that both types of interactions are weighted by the same effective coupling constant $e_r e_s$ which is, actually, a mixed coupling constant. For branes coupled to chiral bosons there is indeed no way to distinguish between magnetic and electric couplings. This is clearly due to the fact that a brane which couples electrically to a chiral boson through $d * F = J$, carries necessarily also a magnetic coupling, because $F = *F$ implies $dF = J$. Therefore an electric brane is automatically turned in a magnetic brane and viceversa. In the case of dyons, considered in the next section, the mutual interactions will exhibit a rather different behaviour.

Another peculiar feature of the above expression for the effective action is represented by the self-interaction of the r-th brane, corresponding to the terms with $r = s$. In the terms describing the Coulomb self-interactions, $J_r \frac{*}{\Box} J_r$, it leads to the standard short distance ultraviolet divergences; but also the mixed self-interactions, $\pm J_r \frac{\delta}{\Box} C_r$, show up ultraviolet divergences. Whereas the formers, being strictly independent on the Dirac-strings, can be regularized in a standard manner, the regularization of the latters has to be handled carefully because it has to be chosen such that the Dirac-anomaly remains an integer multiple of 2π. We will come back to this point in the following subsection.

6.2 Dyons in $D = 4K$

Dyonic branes are branes which carry electric *and* magnetic charge. They can live only in an even dimensional space-time, and in $D = 4K$ they become $(2K-2)$-branes, ($n = 2K - 2 = even$). A system of N such branes is characterized by the charge vectors $e_r^I = (e_r^1, e_r^2)$, $r = 1, \ldots, N$, and by the total currents

$$J^I = \sum_{r=1}^{N} e_r^I J_r \,, \tag{6.8}$$

where the J_r are the Poincaré-duals of the brane worldvolumes. The forms C^I and W^I are expressed as sums completely analogous to (6.8). Notice that also in this case

[7] Notice, however, that (6.6) does not correspond to the most general solution of (6.5).

there is no distinction between branes and dual branes, since they are all $(2K-2)$-branes and carry electric as well as magnetic charge.

The action for such a system is given by (3.12), in terms of two vector potentials $A^I = (A^1, A^2)$, and we have only to specify the signs. In $D = 4K$ we have, in particular,

$$*^2 = \eta = -1\,.$$

The unique difference to the case presented in sections three and four relies in the different parametrizations of the currents, respectively (4.9) and (6.8), and the corresponding differences in the parametrizations of C^I and W^I. We treat now the symmetric and asymmetric theories separately.

Symmetric theory. Keeping the notation of the third line of (3.12) we obtain for the action

$$I_0[A,C,a] = \frac{1}{2}\int \left[F\,\mathcal{P}(v)\,F + dA\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} C\right], \tag{6.9}$$

where now

$$\mathcal{P}(v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} vi_v * + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\left(vi_v - \frac{1}{2}\right). \tag{6.10}$$

Since A^1 and A^2 are now forms of the same degree, A^I can be considered as an SO(2)-doublet. If we consider also the charge vectors e^I_r as SO(2)-doublets, the action I_0 is manifestly invariant under the continuous duality group SO(2). This is due to the fact that the 2×2 matrices appearing in (6.9), the identity and the antisymmetric tensor, are SO(2)-invariant.

Let us now see what happens to the Dirac-anomaly, which triggers the quantum consistency of the system of dyons. Eq. (3.21) gives

$$\mathcal{A}_D = \frac{1}{2}\int W\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} J = \frac{1}{2}\sum_{r,s}(e^1_r e^2_s - e^2_r e^1_s)\int W_r J_s\,. \tag{6.11}$$

The sign flip in $\mathcal{A}_D$, with respect to (3.21), is due to the fact that J^I and W^I are forms of *odd* degree. Since the integrals $\int W_r J_s$ are integers we obtain the quantization conditions

$$\frac{1}{2}(e^1_r e^2_s - e^2_r e^1_s) = 2\pi n_{rs}, \tag{6.12}$$

which are the known Dirac-Schwinger quantization conditions for the charges of dyons in $D = 4K$. The effective action becomes, from (3.27)

$$\begin{aligned}\Gamma[C] &= \frac{1}{2}\int\left[J\frac{*}{\Box}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} J + J\frac{\delta}{\Box}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} C\right] \\ &= \frac{1}{2}\sum_{r,s}\int\left[(e^1_r e^1_s + e^2_r e^2_s)J_r\,\frac{*}{\Box}\,J_s + (e^2_r e^1_s - e^1_r e^2_s)J_r\,\frac{\delta}{\Box}\,C_s\right].\end{aligned} \tag{6.13}$$

The effective coupling constant matrix $\gamma_{rs} = e^1_r e^1_s + e^2_r e^2_s$ for the Coulomb interactions differs now from the one of the mixed interactions, $\beta_{rs} = e^2_r e^1_s - e^1_r e^2_s$, but both matrices are SO(2)-invariant. Notice in particular the absence of mixed self-interactions; this is due to the antisymmetry of the mixed coupling constant matrix β_{rs}. The presence of such interactions will represent the major difference between the symmetric and asymmetric theory.

The generalized Lorentz-force laws can be deduced from $\exp(iI)$. Here I is obtained from I_0 adding the kinetic terms for the dyons, as in (4.3), but now with a single sum. Instead of (4.15) one obtains now[8]

$$T_r \Box\, y^\mu_r = \frac{\partial y^{\nu_1}_r}{\partial \sigma^0} \cdots \frac{\partial y^{\nu_{n+1}}_r}{\partial \sigma^n} \, \left[e^1_r K^2 - e^2_r K^1 \right]^\mu{}_{\nu_{n+1}\cdots\nu_1}(y_r)\,. \tag{6.14}$$

The square bracket reflects again the SO(2)-invariance of the symmetric theory.

Asymmetric theory. The action for this theory is given by

$$\widetilde{I}_0 = I_0 - \frac{1}{2}\int C^1 C^2 = \frac{1}{2}\int \left[F\,\mathcal{P}(v)\,F + dA \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} C - C^1 C^2 \right]. \tag{6.15}$$

The Dirac-anomaly can be read from (3.22)

$$\widetilde{A}_D = \int W^1 J^2 = \sum_{r,s} e^1_r e^2_s \int W_r J_s\,. \tag{6.16}$$

It becomes an integer multiple of 2π if

$$e^1_r e^2_s = 2\pi \widetilde{n}_{rs}\,, \tag{6.17}$$

for each r and s. This is Dirac's original quantization condition and it has to be compared with the Dirac-Schwinger condition (6.12) of the symmetric theory. None of the two conditions implies the other. The most relevant difference is that Dirac's original condition requires quantization also for the electric and magnetic charges of a single dyon: $e^1_r e^2_r = 2\pi \widetilde{n}_{rr}$, which allows for an electric-magnetic self-interaction of the r-th dyon, as we will see in a moment. Moreover, the Dirac-Schwinger conditions admit solutions for the charges with ϑ-angles while Dirac's original condition does not allow for such angles and, to introduce them in the asymmetric theory, one has to amend the action $\widetilde{I}_0$ with a ϑ-term, see section seven.

The principal reason for these differences is that the asymmetric theory looses the SO(2)-duality invariance of the symmetric theory, due to the presence of the additional term $-1/2\int C^1 C^2$ in the action. The unbroken duality sub-group of SO(2) which survives is the discrete group Z_4, generated by

$$\begin{pmatrix} e^1_r \\ e^2_r \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} e^1_r \\ e^2_r \end{pmatrix}. \tag{6.18}$$

[8] Formally one has to sum the right hand sides of (4.15).

Since under this transformation we have $C^1 \to C^2$, $C^2 \to -C^1$, the action changes as[9]

$$\widetilde{I}_0 \to \widetilde{I}_0 + \int C^1 C^2 \,. \tag{6.19}$$

Therefore, strictly speaking $\widetilde{I}_0$ is invariant only under the (trivial) duality group Z_2, generated by minus the identity matrix. However, at the quantum level it is sufficient that the exponentiated action is invariant. Indeed, due to (6.17) and to the fact that the C_r are integer forms, the integral $\int C^1 C^2 = \sum_{r,s} e_r^1 e_s^1 \int C_r C_s$ is an integer multiple of 2π,

The integral $\int C_r C_r$ is, actually, ill-defined because it counts the number of intersections of two "coinciding" surfaces, which is infinite. One can regularize this term by displacing the position of one surface by an arbitrary infinitesimal vector ε^μ orthogonal to the surface. This leads to a framed $(n+2)$-form C_r^ε, and the integral $\int C_r C_r^\varepsilon$ is now well-defined and integer.

Eventually the difference between the two theories is most clearly exhibited by the effective actions. For the asymmetric theory we obtain from (3.30)

$$\begin{aligned}
\widetilde{\Gamma}[C] &= \int \left[\frac{1}{2} J \frac{*}{\Box} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} J + J^2 \frac{\delta}{\Box} C^1 \right] \\
&= \sum_{r,s} \int \left[\frac{1}{2}(e_r^1 e_s^1 + e_r^2 e_s^2) J_r \frac{*}{\Box} J_s + e_r^2 e_s^1 J_r \frac{\delta}{\Box} C_s \right] \\
&= \frac{1}{2} \sum_{r,s} \int \left[(e_r^1 e_s^1 + e_r^2 e_s^2) J_r \frac{*}{\Box} J_s + (e_r^2 e_s^1 - e_r^1 e_s^2) J_r \frac{\delta}{\Box} C_s \right] + \\
&\quad + \sum_r e_r^1 e_r^2 \int J_r \frac{\delta}{\Box} C_r - \frac{1}{2} \sum_{r>s} (e_r^1 e_s^2 + e_r^2 e_s^1) \int C_r C_s \,.
\end{aligned} \tag{6.20}$$

To obtain the last expression we used the Hodge decomposition of the d'alambertian in the form

$$\int \left[J_r \frac{\delta}{\Box} C_s + J_s \frac{\delta}{\Box} C_r \right] = -\int C_r C_s \,, \tag{6.21}$$

which holds for arbitrary even forms $C_{r,s}$ in $D = 4K$ with $J_{r,s} = dC_{r,s}$, thus separating the electric-magnetic interactions in antisymmetric, diagonal and symmetric contributions (w.r.t. r and s). This expression has to be compared with the effective action for the symmetric theory (6.13). The second line in (6.20) equals the effective action for the symmetric theory, the third line represents two additional terms. The second term is made out of integer multiples of π, due to (6.17), and gives in general non vanishing contributions to the (exponentiated) effective action. The first term describes the above mentioned electric-magnetic self-interaction of the r-th dyon,

[9] I_0 is invariant and $C^1 C^2$ changes its sign.

and it needs a regularization. If we use a framing regularization as above, we obtain

$$\Gamma_{\text{self}} \equiv \sum_r e_r^1 e_r^2 \int J_r \frac{\delta}{\Box} C_r^\varepsilon = \frac{1}{2} \sum_r e_r^1 e_r^2 \int \left[J_r \frac{\delta}{\Box} C_r^\varepsilon - J_r^\varepsilon \frac{\delta}{\Box} C_r - C_r C_r^\varepsilon \right], \tag{6.22}$$

where $J_r^\varepsilon = dC_r^\varepsilon$. Here we used again (6.21); notice, however, that in absence of a regularization this formula would have led to the meaningless result $\int J_r \frac{\delta}{\Box} C_r = -1/2 \int C_r C_r$. The first two terms of Γ_{self}, which seem to cancel each other as $\varepsilon \to 0$, converge actually to a finite (non integer) non vanishing result. The third term, on the other hand, is an integer multiple of π and gives a non vanishing contribution to the (exponentiated) effective action, too. The important feature of this regularization is that it keeps the Dirac-anomaly an integer multiple of 2π. Under $\Delta C_r^\varepsilon = dW_r^\varepsilon$, using the definition of Γ_{self} and $\Box = -(d\delta + \delta d)$, we have

$$\Delta\Gamma_{\text{self}} = \sum_r e_r^1 e_r^2 \int J_r W_r^\varepsilon ,$$

which is again a well defined integer multiple of 2π, due to (6.17) for $r = s$.

The major physical effect of this self-interaction term for dyons in $D = 4$ in the asymmetric theory is the spin-statistics transmutation of the r-th dyon from a boson to a fermion, and viceversa, if the integer $\widetilde{n}_{rr}$ appearing in the quantization condition of the asymmetric theory is *odd* [10]. This transmutation does not occur in the symmetric theory. It is instructive to analyse the implementation of the Z_4-invariance of the asymmetric theory a the level of the (regularized) effective action. According to its generator (6.18) we have to perform the replacements $e_r^1 \to e_r^2$, $e_r^2 \to -e_r^1$ and also $C_r^\varepsilon \leftrightarrow C_r$, which implies $J_r^\varepsilon \leftrightarrow J_r$. The second line in (6.20) is manifestly invariant under SO(2) and hence also under Z_4. The second term of the third line changes its sign, but, being an integer multiple of π, it changes by an integer multiple of 2π. The first term of the second line has to be substituted by its regularized version (6.22). Since $e_r^1 e_r^2 \to -e_r^1 e_r^2$ the first two terms in the square bracket compensate this sign change, due to $C_r^\varepsilon \leftrightarrow C_r$, and the third term corresponds to an integer multiple of π and changes by an integer multiple of 2π, as above.

We observe finally that if the quantization conditions (6.12) and (6.17) hold simultaneously, then the difference between the symmetric and asymmetric theories reduces just to Γ_{self}, a part from an integer multiple of 2π. This is due to the trivial identity

$$\frac{1}{2}(e_r^1 e_s^2 + e_r^2 e_s^1) = \frac{1}{2}(e_r^1 e_s^2 - e_r^2 e_s^1) + e_r^2 e_s^1 = 2\pi N_{rs} , \tag{6.23}$$

where we used both quantization conditions.

The generalized Lorentz-force law for the asymmetric theory is identical to the one of the symmetric theory, (6.14), and it is in particular SO(2)-invariant. We would like to stress that the distinction between symmetric and asymmetric theories arises really at the quantum level, at the classical level they are identical.

The analysis of the distinctive features of the symmetric and asymmetric theories performed above, in particular the correct identification of the self-interaction in the asymmetric theory, refines the corresponding analysis performed in [8].

6.3 Dyons in $D = 4K + 2$

This case can be treated in the same way as the preceding one. The branes are $(2K-1)$-branes, ($n = 2K - 1 =$ *odd*), and $*^2 = \eta = +1$. The F's and C's are now *odd* forms.

With respect to the $D = 4K$ case we have only some sign changes in due places. Nevertheless, the main feature of dyons in $D = 4K + 2$ is that the symmetric and asymmetric theories are more similar then the corresponding theories in $D = 4K$. This is due to the fact that, as we will see, in $D = 4K + 2$ the two theories are invariant under the *same* duality group, i.e. $Z_2 \times Z_2'$.

Symmetric theory. The action for the symmetric theory becomes

$$I_0[A, C, a] = \frac{1}{2} \int \left[F\, \mathcal{P}(v)\, F - dA \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} C \right], \tag{6.24}$$

where

$$\mathcal{P}(v) = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} v i_v * + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left(v i_v + \frac{1}{2} \right). \tag{6.25}$$

The sign flip in the last term of (6.24), with respect to (3.12), is due to the fact that dA^I and C^I are here odd forms.

The 2×2 matrices involved in (6.24) are the identity and the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. These matrices are left invariant by the discrete duality group $Z_2 \times Z_2'$, generated by

$$g = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad g' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

respectively. Therefore, the symmetric theory admits the duality-symmetry group $Z_2 \times Z_2'$, the first factor being almost trivial [6].

The Dirac-anomaly is

$$A_D = \frac{1}{2} \int W \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} J = \frac{1}{2} \sum_{r,s} (e_r^1 e_s^2 + e_r^2 e_s^1) \int W_r J_s \,, \tag{6.26}$$

and leads to the quantization condition

$$\frac{1}{2} (e_r^1 e_s^2 + e_r^2 e_s^1) = 2\pi n_{rs} \,, \tag{6.27}$$

first noted in [6], which shows the relative plus sign instead of the minus sign of dyons in $D = 4K$. In particular, we have now also a quantization condition between the charges e_r^1 and e_r^2 of the r-th dyonic brane, $e_r^1 e_r^2 = 2\pi n_{rr}$, allowing for a self-interaction. The effective action is, indeed

$$\begin{aligned}\Gamma[C] &= -\frac{1}{2}\int J\frac{*}{\Box}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} J - J\frac{\delta}{\Box}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} C \\ &= -\frac{1}{2}\sum_{r,s}\int\left[(e_r^1 e_s^1 + e_r^2 e_s^2) J_r \frac{*}{\Box} J_s - (e_r^2 e_s^1 + e_r^1 e_s^2) J_r \frac{\delta}{\Box} C_s\right] \\ &= -\frac{1}{2}\sum_{r,s}\int\left[(e_r^1 e_s^1 + e_r^2 e_s^2) J_r \frac{*}{\Box} J_s\right] + \frac{1}{2}\sum_{r\neq s}\int\left[(e_r^2 e_s^1 + e_r^1 e_s^2) J_r \frac{\delta}{\Box} C_s\right] + \Gamma_{\text{self}}\,,\end{aligned} \tag{6.28}$$

where

$$\Gamma_{\text{self}} = \sum_r \int e_r^1 e_r^2\, J_r \frac{\delta}{\Box} C_r^\varepsilon\,. \tag{6.29}$$

We have again standard Coulomb interactions[10] and mixed interactions weighted by $Z_2 \times Z_2'$-invariant coupling constant matrices. But the main difference w.r.t. the symmetric theory in $D = 4K$ is represented by the presence of the electric-magnetic self-interactions of the dyonic branes, represented by Γ_{self}, which we have regularized as above to preserve the Dirac-anomaly. In $D = 4K + 2$ the identity (6.21) is replaced by

$$\int\left[J_r \frac{\delta}{\Box} C_s - J_s \frac{\delta}{\Box} C_r\right] = -\int C_r C_s\,, \tag{6.30}$$

because the C_r's are here *odd* forms. This allows to decompose the self-interaction as

$$\Gamma_{\text{self}} = \frac{1}{2}\sum_r e_r^1 e_r^2 \int\left[J_r \frac{\delta}{\Box} C_r^\varepsilon + J_r^\varepsilon \frac{\delta}{\Box} C_r - C_r C_r^\varepsilon\right]. \tag{6.31}$$

This time it is the last term which naively would go to zero as $\varepsilon \to 0$, because the C_r are odd forms; but due to the framing it converges to an integer multiple of π which in general is non vanishing.

For what concerns the duality-symmetries, under Z_2 the effective action is trivially invariant, while under the generator g' of Z_2' we have $e_r^1 \leftrightarrow e_r^2$ and $C_r \leftrightarrow C_r^\varepsilon$. The unique term of Γ which transforms non trivially is the last term of Γ_{self} which changes its sign; but since it is an integer multiple of π $\exp(i\Gamma)$ is invariant.

The generalized Lorentz-force equations are

$$T_r \Box\, y_r^\mu = \frac{\partial y_r^{\nu_1}}{\partial \sigma^0} \cdots \frac{\partial y_r^{\nu_{n+1}}}{\partial \sigma^n}\,\left[e_r^1 K^2 + e_r^2 K^1\right]^\mu{}_{\nu_{n+1}\cdots\nu_1}(y_r)\,, \tag{6.32}$$

and they are $Z_2 \times Z_2'$-invariant.

[10]The minus sign of the Coulomb interactions is due to our conventions regarding the Hodge-dual.

Asymmetric theory. The action is

$$\widetilde{I}_0 = I_0 + \frac{1}{2}\int C^1 C^2\,, \tag{6.33}$$

with Dirac-anomaly

$$\widetilde{A}_D = \int W^1 J^2 = \sum_{r,s} e_r^1 e_s^2 \int W_r J_s\,. \tag{6.34}$$

It leads to the quantization conditions

$$e_r^1 e_s^2 = 2\pi \widetilde{n}_{rs}\,, \tag{6.35}$$

which are Dirac's original ones, and they allow for self-interactions, too. The asymmetric action is trivially invariant under Z_2, while under g' it transforms as

$$\widetilde{I}_0 \to \widetilde{I}_0 - \int C^1 C^2\,, \tag{6.36}$$

and $\exp(i\widetilde{I}_0)$ is invariant due to (6.35). The effective action is

$$\begin{aligned}
\widetilde{\Gamma}[C] &= -\int \left[\frac{1}{2} J \frac{*}{\Box} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} J - J^2 \frac{\delta}{\Box} C^1\right] \\
&= \sum_{r,s} \int \left[-\frac{1}{2}(e_r^1 e_s^1 + e_r^2 e_s^2)\, J_r \frac{*}{\Box} J_s + e_r^2 e_s^1\, J_r \frac{\delta}{\Box} C_s\right] \\
&= \Gamma[C] + \frac{1}{2} \sum_{r>s} (e_r^1 e_s^2 - e_r^2 e_s^1) \int C_r C_s\,.
\end{aligned} \tag{6.37}$$

Here $\Gamma[C]$ denotes the effective action of the symmetric theory (6.28), and we used again (6.30). In particular, the self-interaction is present also here, and the unique difference between Γ and $\widetilde{\Gamma}$ is represented by the last sum in (6.37), which amounts to an integer multiple of π. Therefore, also $\exp(i\widetilde{\Gamma})$ is invariant under $Z_2 \times Z_2'$. The generalized Lorentz-force law is the same as for the symmetric theory.

If both quantization conditions (6.27) and (6.35) hold simultaneously, then an argument analogous to (6.23) shows that $\exp(i\Gamma) = \exp(i\widetilde{\Gamma})$, and the two theories become quantum mechanically indistinguishable.

We noted above that in $D = 4$, for the asymmetric theory, the self-interaction leads to a spin-statistics transmutation; the question whether a similar effect occurs also in higher even dimensions, e.g. $D = 6$, is still under investigation. What we have shown here is that *if* it occurs in $D = 4K + 2$, then the effect is present in both type of theories.

Concluding we can say that the dynamics of dyons in an even dimensional space-time can be described at the quantum level by two inequivalent theories. The asymmetric theory requires as consistency condition a Dirac-type quantization condition and the symmetric one a Dirac-Schwinger-type quantization condition. At the quantum level for $D = 4K$ the symmetric theory is SO(2)-invariant and the asymmetric one Z_4-invariant, while for $D = 4K + 2$ both types of theories are invariant under $Z_2 \times Z_2'$.

6.3.1 Chiral factorization

A characteristic property of $(4K + 2)$-dimensional space-times is the existence of chiral and antichiral bosons. This allows to separate a $2K$-form gauge potential in its chiral and antichiral part. Chiral and antichiral bosons should couple respectively to the "chiral" dyonic charges

$$e_r^{\pm} \equiv \frac{1}{2}(e_r^1 \pm e_r^2)\,.$$

We want here to show how this factorization occurs naturally in the covariant action I_0 of the symmetric theory, given in (6.24). It suffices to define

$$A^{\pm} = \frac{1}{2}(A^1 \pm A^2)\,, \qquad C^{\pm} = \frac{1}{2}(C^1 \pm C^2)\,.$$

A short calculation gives then

$$\begin{aligned} I_0[A, C, a] &= \frac{1}{2}\int \left[F\,\mathcal{P}(v)\,F - dA \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} C\right] \\ &= I_0^+[A^+, C^+, a] + I_0^-[A^-, C^-, a]\,, \end{aligned} \tag{6.38}$$

where the actions for chiral bosons $I_0^{\pm}$ are given in (6.3). The effective action factorizes similarly:

$$\Gamma[C] = \Gamma_+[C] + \Gamma_-[C]\,,$$

where $\Gamma_{\pm}[C]$ is given in (6.7).

The action for the asymmetric theory admits only a partial factorization and contains also a mixed term

$$\widetilde{I}_0[A, C, a] = I_0^+[A^+, C^+, a] + I_0^-[A^-, C^-, a] + \int C^- C^+\,.$$

Correspondingly one has $\widetilde{\Gamma}[C] = \Gamma_+[C] + \Gamma_-[C] + \int C^- C^+$. There remains, however, a link between chiral and antichiral currents since the quantization conditions (6.27) and (6.35), in terms of the chiral charges, read

$$\begin{aligned} e_r^+ e_s^+ - e_r^- e_s^- &= 2\pi n_{rs} \qquad \text{(symmetric)} \\ e_r^+ e_s^+ - e_r^- e_s^- + e_r^- e_s^+ - e_r^+ e_s^- &= 2\pi \widetilde{n}_{rs} \qquad \text{(asymmetric)}\,, \end{aligned} \tag{6.39}$$

and mix up chiral with antichiral charges. These conditions are less restrictive then the ones for chiral branes, i.e. $e_r^{\pm} e_s^{\pm} = 2\pi n_{rs}$, and admit therefore more solutions. The case of chiral branes corresponds to $e_r^- = 1/2(e_r^1 - e_r^2) = 0 = C^-$, $e_r^+ = e_r^1$, meaning that the electric charges equal the magnetic ones. In this case the antichiral boson A^- decouples, it becomes a free field, and one recovers the quantization conditions (6.5) for chiral (self-dual) branes.

The invariant field strengths corresponding to the chiral and antichiral bosons are given by $K^\pm = 1/2(K^1 \pm K^2)$ and satisfy

$$*K^\pm = \pm K^\pm \,.$$

7. More general couplings

In this section we present generalizations of the PST-action which describe couplings to additional fields, and address the issue of ϑ-angles.

We recall that for all models discussed in this paper, except for chiral bosons, equivalent formulations in terms of a single gauge potential are obtained by dropping in (3.12) in the first line the term $f^1 * f^1$, or in the second line the term $f^2 * f^2$. In the first case one obtains a formulation in terms of the potential A^1, and in the second case a formulation in terms of the (dual) potential A^2. The principal drawback of formulations with a single gauge potential is that duality symmetries are not manifest; they can be realized only through non-local transformations [8].

Formally these formulations can be obtained from the PST-formulation in the following way. First one has to S-dualize the scalar auxiliary field a to an auxiliary $(D-2)$-form a_{D-2}, obtaining thus the so called "dual" PST-action [11], again in terms of the two potentials A^1 and A^2, $I_0^{\rm dual}[A^I, C, a_{D-2}]$. Then one has to perform the functional integration over A^2 (or A^1) of $\exp(iI_0^{\rm dual})$. After this functional integration the field a_{D-2} decouples, and one obtains the Schwinger-like action in terms of a single gauge potential (for more details see [8]).

Gravitation. The dynamics of the interacting system branes/potentials described in the preceding sections was assumed to occur in a background with metric $G_{\mu\nu}(x)$. Implicitly all indices were raised and lowered with this metric. Diffeomorphism invariance is achieved thanks to the manifest Lorentz-invariance of the PST-method in the flat case: the minimal-coupling-recipe gives therefore rise to manifestly diffeomorphism invariant actions. This is one of the fundamental advantages of the PST-approach. Moreover, the Dirac-anomaly is a topological invariant i.e. metric independent, and so the relevant quantization conditions ensure Dirac-string independence of the exponentiated actions also in the presence of gravitation. To introduce a dynamical gravitational field it is therefore sufficient to add the Einstein-Hilbert action.

Coupling to other fields. The PST-action admits also consistent couplings of the system gauge-potentials/branes to other fields. These couplings are represented in general by a couple of p^I-forms L^I. $L^1(L^2)$ couples electrically to $A^2(A^1)$ and magnetically to $A^1(A^2)$. In supergravity theories, for example, the (composite) fields L^I correspond to Chern-Simons forms, made out of other p-form potentials or of anomaly-cancelling terms, or to bilinears in the fermions. The modified dynamics is then expressed by the equations

$$\begin{aligned} H^I &\equiv F^I + L^I = dA^I + C^I + L^I \\ H^1 &= *H^2 \,. \end{aligned} \tag{7.1}$$

Since we have now $dH^I = J^I + dL^I$, the forms dL^I must be (gauge) invariant forms. Actually, the fields L^I can depend also on the brane worldvolumes themselves; in this case the L^I are required to be invariant under Dirac-string changes.

The action which takes the new couplings consistently into account is given for the symmetric theory by

$$\begin{aligned} I_L &= I_0[A, C+L, a] + \eta\frac{1}{2}\int \left(L^1C^2 - C^1L^2\right) \\ &= \frac{1}{2}\int \left[H\,\mathcal{P}(v)\,H + \eta\left((C^1+L^1)dA^2 - dA^1(C^2+L^2) + L^1C^2 - C^1L^2\right)\right]. \end{aligned} \tag{7.2}$$

The replacement $C \to C+L$ in the PST-action is required to cope with the PST-symmetries; this replacement has to be made also in the transformation laws for A^I in (3.17). The additional term $\eta 1/2\int (L^1C^2 - C^1L^2)$ is needed to preserve the Dirac-anomaly. This can be seen from the the second line of (7.2): under a change of the Dirac-string the variation of the last term cancels the variation of $L^1dA^2 - dA^1L^2$. Since the H^I are invariant under string changes we have then $\Delta I_L = A_D = \Delta I_0$, i.e. the Dirac-anomaly in the absence of the composite fields L^I. Therefore, string-independence is again ensured by the "old" quantization conditions.

In Schwinger-like formulations the same recipe works; one has to replace $C \to C+L$, and to add the term $\eta 1/2\int (L^1C^2 - C^1L^2)$. For the asymmetric theory one has to add, instead of this terms, the term $-\eta\int C^1L^2$. Eventually this leads for the asymmetric theory to the action

$$\widetilde{I}_L = I_L + \frac{1}{2}\eta\int C^1C^2. \tag{7.3}$$

In the usual gauge for (3.16), the equations of motion for the coupled system become

$$\begin{aligned} T_r^2\Box^{(n)}y_r^\mu &= \eta(-)^{nD}\,e_r^2\,\frac{\partial y_r^{\nu_1}}{\partial\sigma^0}\cdots\frac{\partial y_r^{\nu_{n+1}}}{\partial\sigma^n}\,(F^1)^\mu{}_{\nu_{n+1}\cdots\nu_1}(y_r) \\ T_r^1\Box^{(D-n-4)}y_r^\mu &= \eta(-)^{D+n+1}\,e_r^1\,\frac{\partial y_r^{\nu_1}}{\partial\sigma^0}\cdots\frac{\partial y_r^{\nu_{D-n-3}}}{\partial\sigma^{D-n-4}}\,(F^2)^\mu{}_{\nu_{D-n-3}\cdots\nu_1}(y_r) \\ F^1 + L^1 &= *(F^2 + L^2)\,. \end{aligned}$$

Notice that the field strengths appearing in the Lorentz-force law are still the F's and not the H's. The interaction between the branes and the fields L^I is thus introduced indirectly through the modified duality relation between F^1 and F^2.

ϑ-angles for dyons. Dyons admit generalized couplings with ϑ-angles only in $4K$-dimensional space-times. In $D = 4K + 2$ one can introduce ϑ-terms if one doubles the degrees of freedom of the gauge fields. We treat here only the former case and refer the reader for the latter case to reference [6].

In the symmetric theory of dyons in $D = 4K$ ϑ-angles are already present. This is due to the particular form of the relevant Dirac-Schwinger quantization conditions (6.12), with the minus sign. Apart from an SO(2)-rotation, their general solutions can be parameterized as[11]

$$\begin{aligned} e_r^1 &= \frac{2\pi}{e_0} m_r \\ e_r^2 &= \frac{n_r}{L} e_0 + \vartheta \frac{2\pi}{e_0} m_r\,, \end{aligned} \tag{7.4}$$

where m_r, n_r and L are integers restricted to

$$\frac{1}{2}(n_r m_s - n_s m_r) = 0 \bmod L\,. \tag{7.5}$$

These solutions show up the shift of say the electric charge by an amount proportional to the magnetic charge, which is characteristic for ϑ-angles. The integer L can in general not be set equal to 1.

The asymmetric theory requires the Dirac-conditions (6.17), which have the general solutions

$$\begin{aligned} e_r^1 &= \frac{2\pi}{e_0} m_r \\ e_r^2 &= n_r e_0\,, \end{aligned} \tag{7.6}$$

and do not allow for ϑ-angles. To introduce them one has to modify the action $\widetilde{I}_0$ as follows

$$\begin{aligned} \widetilde{I}_0^\vartheta &\equiv I_0[A, C^1, C^2 + \vartheta C^1, a] - \frac{1}{2}\int C^1 C^2 \\ &= \int \frac{1}{2}\left(F^1 * F^1 + f_\vartheta^1 * f_\vartheta^1\right) + dA^1\left(C^2 + \vartheta C^1\right) + \frac{1}{2}\vartheta C^1 C^1 \\ &= \int \frac{1}{2}\left(F^1 * F^1 + f_\vartheta^1 * f_\vartheta^1\right) + dA^1 C^2 + \frac{\vartheta}{2} F^1 F^1. \end{aligned} \tag{7.7}$$

Notice that in the definition of $\widetilde{I}_0^\vartheta$ the shift $C^2 \to C^2 + \vartheta C^1$ has not been performed in the last term, because otherwise we would have obtained simply a (trivial) redefinition of the charges. In the second line above we used the first expression for I_0 in (3.12), where f_ϑ^1 is obtained from f^1 with the replacement $C^2 \to C^2 + \vartheta C^1$. In

[11] For a derivation of the solutions see [8]; w.r.t. this reference we rescaled the charge e_0 and reshuffled a factor of 2.

the third line the ϑ-term shows up in the typical way as $\int F^1 F^1$ which is, here, not a topological term because $dF^1 \neq 0$. The formulation in terms of a single potential is obtained dropping the term $\int f^1_\vartheta * f^1_\vartheta$.

Since F^1 is string-independent, the Dirac-anomaly remains the old one, $\widetilde{A}_D$ in (6.16), the quantization conditions are again Dirac's original ones (6.17), and the "nominal" individual charges e^I_r are still of the form (7.6). But since the Bianchi identities and equations of motion are now

$$\begin{aligned} dF^1 &= J^1 \\ dF^2 &= J^2 + \vartheta J^1 \\ F^1 &= *F^2, \end{aligned} \tag{7.8}$$

the physical individual charges are given by

$$\begin{aligned} E^1_r &= \frac{2\pi}{e_0} m_r \\ E^2_r &= n_r e_0 + \vartheta \frac{2\pi}{e_0} m_r\,, \end{aligned} \tag{7.9}$$

where m_r and n_r are *arbitrary* integers. These formulae have to be compared with (7.4), and show once more the inequivalence between the symmetric and asymmetric theory.

For what concerns the effective action it is worthwhile to notice that the ϑ-angle affects only the Coulomb-interaction in (6.20), through $C^2 \to C^2 + \vartheta C^1$, while it drops out from all electric-magnetic interactions, the last ones being protected by the Dirac-anomaly, as shown in [8].

8. Heterotic strings and five-branes in $D = 10$: cancellation of anomalies revisited

The classical Green-Schwarz anomaly cancellation mechanism [12] applies, as it stands, to $N = 1$, $D = 10$ supergravity; it ensures also the cancellation of two-dimensional anomalies in the SO(32)-heterotic string sigma-model, in a supergravity background. For the SO(32)-heterotic five-brane, which still waits for a κ-symmetric sigma-model action, the Green-Schwarz mechanism has to be amended by the so called "inflow anomaly cancellation mechanism" [13]. In this section we show how this mechanism can be implemented through a bosonic action when a heterotic string and a heterotic five-brane are simultaneously present and interact with a dynamical $N = 1$, $D = 10$ bosonic supergravity background. For simplicity we omit the dilaton and set charges and tensions to unity.

Following the conventions of the text we describe the two-dimensional worldvolume of the string Σ_2 by an eight-form $J^2 \equiv J_8$ and the six-dimensional worldvolume of the fivebrane Σ_6 by a four-form $J^1 \equiv J_4$. Dirac-strings are introduced through $J_8 = dC_7$ and $J_4 = dC_3$. The corresponding supergravity potentials are conventionally the two- and six-forms $A^1 \equiv B_2$ and $A^2 \equiv B_6$, which are dual to each other.

We recall now briefly the structure of the anomalies carried by the system. The local symmetry groups involved are the Lorentz group SO(1, 9), the gauge group SO(32) $\otimes$ SU(2), and the structure group of the normal bundle of the five-brane SO(4); we call the corresponding two-form curvatures respectively R, $F \oplus G$ and T. The structure group of the normal bundle of the string SO(8) does not enter the game since it is anomaly free. The string is neutral with respect to SU(2), while the chiral fermions living on the fivebrane carry non trivial representations of SO(32) $\otimes$ SU(2). The correct field content of the five-brane has been guessed in [14], and this allowed recently to determine its total anomaly polynomial [15]. We report here the anomaly polynomials for respectively ten-dimensional supergravity I_{12}, heterotic five-branes I_8, and heterotic strings I_4: [12]

$$\begin{aligned} I_{12} &= X_4\, X_8\,, \\ I_8 &= X_8 + (X_4 + \chi_4)\, Y_4\,, \\ I_4 &= X_4\,, \end{aligned}$$

where

$$\begin{aligned} X_8 &= \frac{1}{192(2\pi)^4}\left(\mathrm{Tr}\, R^4 + \frac{1}{4}(\mathrm{Tr}\ R^2)^2 - \mathrm{Tr}\ R^2\ \mathrm{Tr}\ F^2 + 8\ \mathrm{Tr}\ F^4\right) \\ X_4 &= \frac{1}{4(2\pi)^2}\left(\mathrm{Tr}\ R^2 - \mathrm{Tr}\ F^2\right) \\ Y_4 &= \frac{1}{48(2\pi)^2}\left(\mathrm{Tr}\ R^2 - 2\ \mathrm{Tr}\ T^2 - 24\ \mathrm{Tr}\ G^2\right) \\ \chi_4 &= \frac{1}{8(2\pi)^2}\,\varepsilon^{a_1a_2a_3a_4}\, T^{a_1a_2}\, T^{a_3a_4}\,. \end{aligned}$$

For each invariant polynomial we introduce standard descent polynomials according to

$$\begin{aligned} X_8 &= dX_7 \\ \delta X_7 &= dX_6\,, \end{aligned} \tag{8.1}$$

and we use an analogous notation for the others. The polynomial χ_4 denotes the Euler class of the five-brane normal bundle and $a_1 = (1, \ldots, 4)$.

It should be stressed that the polynomials formed with R and F are pull-backs of ten-dimensional differential forms, i.e forms which are defined in the whole target space, while the ones involving T and G live strictly on the five-brane worldvolume. Therefore the string polynomial I_4, being a pull-back, can be cancelled by the ordinary Green-Schwarz mechanism while, for the five-brane, only the term X_8 in I_8 can

[12]The actual polynomials have to be multiplied by a factor of 2π, $\widehat{I}_n = 2\pi I_n$.

be cancelled in this way. To be more precise, an equation of the form $dH_7 = I_8$ does not make sense since H_7 is a target-space form while I_8 is not. This is precisely the problem faced, and solved, by the inflow mechanism.

The anomaly cancelling terms are written usually as

$$I_{\rm anom} = \begin{cases} \int B_2 X_8 & \text{for } D=10 \text{ supergravity} \\ \int B_2 J_8 = \int_{\Sigma_2} B_2 & \text{for the string} \\ \int (B_6 + B_2 Y_4) J_4 = \int_{\Sigma_6} (B_6 + B_2 Y_4) & \text{for the five-brane,} \end{cases} \tag{8.2}$$

where the anomalous transformation laws for the potentials are [15, 16] $\delta B_6 = -X_6$, $\delta B_2 = -X_2$, but also $\delta B_2^{(0)} = -\chi_2$ (see below for the explanation). Here $B_2^{(0)}$ denotes the pull-back of B_2 on the five-brane, and χ_2 is associated to χ_4 through the descent formalism. With these choices the above counterterms cancel indeed the anomalies associated through the descent formalism to I_{12}, I_4 and I_8 respectively. However, the last transformation law presents a problematic feature: while in the target space B_2 transforms by $-X_2$, when restricted to the five-brane it should transform also by $-\chi_2$. But in the sense of distributions, from the point of view of the target space field B_2, the last transformation is, actually, zero. On the other hand, if one considers this transformation as non trivial, thus cancelling the five-brane anomaly, then the term $-\chi_2$ should affect also the variation of the counterterm of the supergravity sector.

Without pursuing the discussion of this point, let us give now the action which follows from our general framework and which avoids, in particular, these ambiguities. First of all we postulate for the invariant curvatures the Bianchi-identities

$$\begin{aligned} dH_3 &= J_4 + X_4 \\ dH_7 &= J_8 + X_8 + Y_4 J_4 \,, \end{aligned} \tag{8.3}$$

and the equation of motion

$$H_7 = *H_3 \,. \tag{8.4}$$

The Bianchi identity for H_3 equals a target space form to a target space form. The one for H_7 involves at the r.h.s. the eight-form $Y_4 J_4$. Y_4 is a four-form which lives strictly on the five-brane, but it is multiplied by the target-space form J_4 which is essentially the δ-function on the five-brane, so the product is again a *target-space* eight-form.

Actually, the expression $Y_4 J_4$ is formal, because in general the product between a target-space form and a form living only on the five-brane *is not* a target-space form. To be more precise, by definition with $Z_8 \equiv Y_4 J_4$ we mean the target-space eight-form whose components are

$$Z_{\mu_1 \cdots \mu_8}(x) = \frac{1}{2!4!}\, \varepsilon_{\mu_1 \cdots \mu_8 \nu_1 \nu_2} \int_{\Sigma_6} d^6\sigma\, \varepsilon^{i_1 \cdots i_6} \partial_{i_1} y^{\nu_1} \partial_{i_2} y^{\nu_2} Y_{i_3 i_4 i_5 i_6}\, \delta^{10}(x - y(\sigma)) \,, \tag{8.5}$$

where the five-brane is parametrized by $y^\mu(\sigma)$, and $Y_{i_3 i_4 i_5 i_6}(\sigma)$ are the components of the five-brane four-form Y^4.

The short-hand (factorized) notation $Y_4 J_4$ is eventually justified by the important fact that the Leibnitz-rule for the differential holds still true:

$$d(Y_4 J_4) = Y_4 dJ_4 + dY_4 J_4 \,.$$

In the present case the r.h.s is zero, but this formula would hold also if Y_4 were not closed and if the worldvolume of the five-brane were a manifold with boundary. In this more general case the two terms on the r.h.s. of the formula are defined in complete analogy with (8.5) and they represent again a well defined target-space form. For this reason we continue to use the compact notation of (8.3).

From the Bianchi identities it is now easy to reconstruct potentials

$$\begin{aligned} H_3 &= dB_2 + C_3 + X_3 \\ H_7 &= dB_6 + C_7 + X_7 + Y_3 J_4 \,. \end{aligned} \tag{8.6}$$

In this formula the term $Y_3 J_4$ is defined in complete analogy to (8.5), and Y_3 is the Chern-Simons form associated to $Y_4 = dY_3$.

Requiring that the curvatures are invariant under the local symmetries at hand, we obtain the transformation laws for the potentials:

$$\begin{aligned} \delta B_2 &= -X_2 \\ \delta B_6 &= -X_6 - Y_2 J_4 \,, \end{aligned} \tag{8.7}$$

which are now free from ambiguities since δB_6 represents a well-defined distribution. The additional term which is present in the definition of H_7 is supported on the five-brane worldvolume and it is invariant under Dirac-string changes. This means that H_7 and H_3 are invariant under string changes with the *standard* transformation laws for the potentials:

$$\begin{aligned} \Delta C_7 &= dW_6, \qquad \Delta B_6 = -W_6 \\ \Delta C_3 &= dW_2, \qquad \Delta B_2 = -W_2 \,. \end{aligned} \tag{8.8}$$

The system of equations we got is now precisely of the form (7.1), with $L^1 \equiv L_3 = X_3$ and $L^2 \equiv L_7 = X_7 + Y_3 J_4$. It is then straightforward to apply our recipe of the preceding section to write the corresponding action. As we said, for the case branes/dual branes the appropriate theory is the asymmetric one. Including also the graviton and the kinetic terms for the branes it is given by

$$I = \int_{\Sigma_{10}} \sqrt{G}\, R + \int_{\Sigma_2} \sqrt{g_2} + \int_{\Sigma_6} \sqrt{g_6} + \widetilde{I}_L \,.$$

The (asymmetric) action $\widetilde{I}_L$ represents the interaction between branes and potentials. It admits a PST-version in terms of B_6 *and* B_2 (see (7.3) and (7.2), a Schwinger-like version in terms of only B_2 and one in terms of only B_6. The comparison between them is instructive, so we write them all explicitly:[13]

$$\begin{aligned}
\widetilde{I}_L[B_2,B_6] &= -\frac{1}{2}\int_{\Sigma_{10}} [H\,P(v)\,H + (X_7+Y_3J_4+C_7)dB_2 + (X_3+C_3)dB_6 + \\
&\qquad +(X_7+Y_3J_4)C_3 + X_3C_7 + C_3C_7] \\
\widetilde{I}_L[B_2] &= -\int_{\Sigma_{10}} \left[\frac{1}{2}H_3 * H_3 + (X_7+Y_3J_4+C_7)dB_2 + (X_7+Y_3J_4)C_3\right] \\
\widetilde{I}_L[B_6] &= -\int_{\Sigma_{10}} \left[\frac{1}{2}H_7 * H_7 + (X_3+C_3)dB_6 + X_3C_7\right].
\end{aligned}$$

Their Dirac-anomalies can be computed using (8.8),

$$\begin{aligned}
\Delta\widetilde{I}_L[B_2,B_6] &= -\int_{\Sigma_{10}} W_2J_8 \\
\Delta\widetilde{I}_L[B_2] &= -\int_{\Sigma_{10}} W_2J_8 \\
\Delta\widetilde{I}_L[B_6] &= -\int_{\Sigma_{10}} W_6J_4\,,
\end{aligned}$$

and they are all integer.

Eventually the form of all these actions is completely determined by the requirements: 1) the equations of motion have to be (8.3) and (8.4); 2) their Dirac-anomalies have to be integers (in this section we have set the charges to unity).

It remains to compute the anomalies of these actions under the local symmetries and to check whether they cancel the quantum anomalies. Taking the above transformations of the potentials into account the evaluation of the variations is a mere exercise and leads to

$$\begin{aligned}
\delta\widetilde{I}_L[B_2,B_6] &= -\frac{1}{2}\int_{\Sigma_{10}}(X_2X_8+X_4X_6) - \int_{\Sigma_2} X_2 - \\
&\quad - \int_{\Sigma_6}\left(X_6 + \frac{1}{2}(X_2Y_4+Y_2X_4) + Y_2J_4\right) \\
\delta\widetilde{I}_L[B_2] &= -\int_{\Sigma_{10}} X_2X_8 - \int_{\Sigma_2} X_2 - \int_{\Sigma_6}(X_6+X_2Y_4+Y_2J_4) \\
\delta\widetilde{I}_L[B_6] &= -\int_{\Sigma_{10}} X_4X_6 - \int_{\Sigma_2} X_2 - \int_{\Sigma_6}(X_6+Y_2X_4+Y_2J_4)\,.
\end{aligned}$$

The anomaly supported on the string worldsheet corresponds in each case to the polynomial $-I_4$. The same happens for the anomaly supported on the target-space, corresponding to $-I_{12}$, which however comes out in three different but cohomologically equivalent cocycles. The cocycle produced by the PST-action is symmetrized

[13] For the additional overall minus sign see the note 2.

in $X_4 \leftrightarrow X_8$, reflecting the fact that the action involves B_2 and B_6 in a symmetric way. The anomaly supported on the five-brane contains in each case the term

$$-\int_{\Sigma_6} Y_2 J_4 = -\int_{\Sigma_{10}} Y_2 J_4 J_4 \,, \tag{8.9}$$

which has to be properly defined. The product of target-space forms $J_4 J_4$ is indeed ill-defined, because it contains the square of a δ-function, and amounts formally to $0 \cdot \infty$. If one of the two factors were a regular form, say K_4, then the other factor J_4 would perform simply the pull-back of K_4 on the five-brane worldvolume Σ_6. If we indicate this pull-back form with $K_4^{(0)}$, then one has the equality

$$J_4 K_4 = J_4 K_4^{(0)}\,;$$

since $K_4^{(0)}$ is now a form on the five-brane worldvolume the r.h.s. of this equation is defined as in (8.5). For our original product this procedure would lead to the form $J_4^{(0)}$ which is, however, again ill-defined. In the literature one performs usually the *cohomological* identification [16, 17]

$$J_4^{(0)} \to \chi_4\,,$$

where χ_4 is the euler class. Actually, this identification can be realized at the level of *differential forms*. For a flat metric, $G_{\mu\nu}(x) = \eta_{\mu\nu}$, it can be realized in a very direct way as follows. [14] Let us go back to the definition of J_4 as a four-current, more precisely to its explicit expression given in the appendix C in formula (C.1), with $p = 6$ and $D = 10$. We regularize this formula by choosing a gaussian approximation for the δ-function appearing there,

$$\delta^{10}(x-y) \to \delta^{10}_\varepsilon(x-y) \equiv \frac{e^{-\eta_{\mu\nu}(x-y)^\mu(x-y)^\nu/\varepsilon}}{(\pi\varepsilon)^5}\,.$$

For $\varepsilon \to 0$ we have $\delta^{10}_\varepsilon(x-y) \to \delta^{10}(x-y)$, in the sense of distributions. This regularization gives rise to a smooth current J_4^ε, the product $J_4 J_4^\varepsilon$ is now well defined and one has the pull-back equality $J_4 J_4^\varepsilon = J_4 (J_4^\varepsilon)^{(0)}$. The limit for $\varepsilon \to 0$ can now be evaluated explicitly [7] and the result is indeed

$$\lim_{\varepsilon\to 0} J_4 J_4^\varepsilon = \lim_{\varepsilon\to 0} J_4 (J_4^\varepsilon)^{(0)} = J_4\, \chi_4\,.$$

More precisely, it can be shown that one has the local *pointwise* limit on the five-brane

$$\lim_{\varepsilon\to 0} (J_4^\varepsilon)^{(0)} = \chi_4\,.$$

[14]Notice that even for a flat target-space metric the normal curvature $T^{a_1 a_2}$, composing χ_4, is in general non vanishing due to the *intrinsic* curvature of the five-brane.

This allows to define the term in eq. (8.9) through our limiting procedure as

$$-\lim_{\varepsilon\to 0}\int_{\Sigma_{10}} Y_2 J_4 J_4^\varepsilon = -\lim_{\varepsilon\to 0}\int_{\Sigma_{10}} Y_2 J_4 (J_4^\varepsilon)^{(0)} = -\int_{\Sigma_{10}} Y_2 J_4 \chi_4 = -\int_{\Sigma_6} Y_2\, \chi_4\,.$$

With this specification also the anomalies supported on the five-brane arise from the anomaly polynomial $-I_8$, and each of the three actions given above cancels the quantum anomalies, a part from trivial cocycles.

We remark that a more sophisticated regularization of J_4, in a curved target-space, can be achieved by means of the theory of characteristic currents, developed in [18]. This regularization enjoies the property that, in addition to $\lim_{\varepsilon\to 0} J_4^\varepsilon = J_4$, one has

$$(J_4^\varepsilon)^{(0)} = \chi_4\,,$$

for every ε.

The main difference between our framework, based on a systematic introduction of Dirac-strings, and the usual way of presenting the inflow mechanism, is given by the transformation laws of the potentials (8.7), which are common to the three actions: B_2 transforms as in ordinary source-free $D = 10$ supergravity, and δB_6 carries an additional term localized at the five-brane, which represents a well defined target-space form. The difference arises also in the mechanism which produces the anomalies. For example, making reference to the Schwinger-like action for B_2, the anomaly $-\int_{\Sigma_6} Y_2\, \chi_4$ is obtained in the usual framework, based on $\delta B_2^{(0)} = -\chi_2$, by varying the term $-\int_{\Sigma_{10}} Y_3 J_4 dB_2 = \int_{\Sigma_6} Y_4 (B_2)^{(0)}$; in our framework it is obtained by varying the term $-\int_{\Sigma_{10}} Y_3 C_3 J_4$, whose presence is implied by the requirement of unobservability of the Dirac-string.

Another peculiar feature of our framework is provided by a particular interplay between Dirac-anomalies and gauge-anomalies. One could cancel the heterotic string anomaly of the classical action, $-\int_{\Sigma_2} X_2$, by adding the term $\int_{\Sigma_{10}} X_3 C_7$; similarly one could cancel the X_8-part of the five-brane anomaly by adding the term $\int_{\Sigma_{10}} X_7 C_3$. But these counterterms would then produce non integer Dirac-anomalies.

If no heterotic string (five-brane) is present, which corresponds to drop C_7 (C_3), then the Schwinger-like action for B_6 (B_2) can be written without introducing the Dirac-string C_3 (C_7), since $\int_{\Sigma_{10}} C_3 dB_6 = -\int_{\Sigma_6} B_6$, and similarly for the other case. If strings and five-branes are simultaneously present, for Schwinger-like actions the introduction of at least one Dirac-string (C_3 or C_7) is unavoidable, for the PST-action one must introduce both.

Let us finally comment on a problem which we overlooked until now, and which concerns the term

$$-\int_{\Sigma_{10}} Y_3 C_3 J_4\,, \tag{8.10}$$

which is present in the PST-action, with a factor of 1/2, and in the Schwinger-like action for B_2. This term, as it stands, is again ill-defined because the five-brane worldvolume, represented by J_4, intersects its own Dirac-string, represented by C_3, on its boundary. This leads again to a δ^2-like singularity in $C_3 J_4$, as happened for $J_4 J_4$. Because of this singularity the manipulations we performed in computing the gauge anomalies were formal. Handling it in the same way as the one encountered above i.e. through a gaussian regularization, one can eventually show that the correct definition of the term is

$$-\int_{\Sigma_{10}} Y_3\, C_3\, J_4 \to -\int_{\Sigma_{10}} Y_3\, \widetilde{\chi_3}\, J_4 = -\int_{\Sigma_6} Y_3\, \widetilde{\chi_3}\,,$$

where $\widetilde{\chi_3}$ is an "invariant" Chern-Simons form for the Euler class, $d\widetilde{\chi_3} = \chi_4$. It differs from the standard Chern-Simons form by an exact differential, $\widetilde{\chi_3} = \chi_3 + d\Psi_2$. With this specification the gauge anomalies (and Dirac-anomalies) can be evaluated without ambiguity and the results coincide with the ones stated in the text.

An explicit expression for $\widetilde{\chi_3}$ and its geometric interpretation will be furnished in [7], where we will present further applications of our framework, especially to the M-theory five-brane.

A. Proof of the invariance of the action

The invariance of I_0 under the PST-symmetries (3.15)- (3.17) can be inferred easily from the form of its variation under general transformations of A^I and a:

$$\delta I_0 = (-)^n \eta \int \left(v f^1 d\delta A^2 + (-)^{(D+1)(n+1)} v f^2 d\delta A^1 + \frac{1}{\sqrt{-(\partial a)^2}} v f^1 f^2 d\delta a \right).$$

This formula allows also to deduce the equations of motion for A^I and a, respectively $d(vf^I) = 0$ and $d\left(\frac{v}{\sqrt{-(\partial a)^2}}\, f^1 f^2\right) = 0\,.$

B. Derivation of the effective action

Taking the gauge-fixings of the PST-symmetries into account and disregarding for the moment the C-independent normalization, (3.26) is written as

$$e^{i\Gamma[C]} \equiv \int \{\mathcal{D}A\}_{gf}\, \{\mathcal{D}a\}\; e^{iI_0[A,C,a]}\, \delta(a - a_0)\,. \tag{B.1}$$

The gauge-fixings of the symmetries (3.15) and (3.16) have still to be specified.

The evaluation of the functional integral can be most easily performed if one notes the following identity

$$I_0[A,C,a] = \Gamma[C] + \frac{1}{2}\int H \Lambda(v) * H\,. \tag{B.2}$$

$\Gamma[C]$ is defined as in (3.27), $H \equiv F^1 - *F^2$, and $\Lambda(v)$ is an operator which sends p^1-forms in p^1-forms,

$$\Lambda(v) = \eta(-)^{D+1}\frac{d\delta}{\Box} + (-)^{D+n} v i_v \,.$$

The functional integral over a sets $v = v(a_0) \equiv v_0$ and the r.h.s. of (B.1) becomes

$$e^{i\Gamma[C]} \cdot \int \{\mathcal{D}A\}_{gf}\, e^{\frac{i}{2}\int H\Lambda(v_0)*H}. \tag{B.3}$$

The C-dependence in the remaining functional integral is spurious and can be eliminated through the shifts

$$A^1 \to A^1 + (-)^D * \frac{d}{\Box}(C^2 - \eta * C^1)$$
$$A^2 \to A^2 + (-)^D \eta * \frac{d}{\Box}(C^1 - *C^2)\,,$$

which imply

$$H \to dA^1 - *dA^2\,.$$

These shifts are compatible, for example, with the linear gauge fixings for the symmetries (3.15), (3.16)

$$i_v A^I = 0 = d * A^I\,.$$

The remaining integral in (B.3) gives, therefore, a C-independent constant which cancels precisely against the normalization. This can be seen using the identity (B.2) with $C^I = 0$. The result is (3.27), q.e.d.

C. Integration of p-forms and Poincaré-duality

We give here explicit coordinate representations for the Poincaré-dual of a generic hypersurface. We normalize the integral of a generic p-form Ψ_p on a p-dimensional hypersurface Σ_p, parameterized by $y^\mu(\sigma^i)$, $i = (1, \ldots, p)$, as follows

$$\int_{\Sigma_p} \Psi_p = \int d^p\sigma\, \frac{\partial y^{\mu_1}}{\partial \sigma^1} \cdots \frac{\partial y^{\mu_p}}{\partial \sigma^p}\, \Psi_{\mu_p \cdots \mu_1}\,.$$

The Poincaré-dual to Σ_p, a $(D-p)$-form, admits then the coordinate representation

$$\Phi_{\Sigma_p}(x) = \frac{(-1)^{D+1}}{(D-p)!}\, dx^{\mu_1} \cdots dx^{\mu_{D-p}}\, \varepsilon_{\mu_1 \cdots \mu_{D-p}\nu_1 \cdots \nu_p} \int d^p\sigma\, \delta^D(x - y(\sigma))\, \frac{\partial y^{\nu_1}}{\partial \sigma^1} \cdots \frac{\partial y^{\nu_p}}{\partial \sigma^p}\,. \tag{C.1}$$

It satisfies (4.5) for every p-form Ψ_p. The formula (C.1) gives explicit coordinate expressions for the currents J^I when Σ_p is the brane worldvolume and the integration over the σ^i covers the whole worldvolume. If the worldvolume is boundaryless, as supposed in the text, one can introduce a $(p+1)$-dimensional hypersurface parameterized, for example, by $y^\mu(\sigma^i, s)$, with $0 \le s \le \infty$, such that $y^\mu(\sigma^i, 0) = y^\mu(\sigma^i)$. Then the representations for the C^I are again of the form (C.1), with $p \to p+1$, where the integral over the new coordinate s is restricted to the interval $[0, \infty]$.

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